A CRITICAL STUDY OF

BRAHMAGUPTA AND HIS WORKS

(A most distinguished Indian Astronomer and Mathematician of the sixth century A D)

By

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THE
INDIAN INSTITUTE
OF
ASTRONOMICAL & SANSKRIT RESEARCH
NEW DELHI

Published by :

Shri Ram Swarup Sharma

Director :

The Indian Institute of Astronomical and Sanskrit Research.
26/95 West Patel Nagar, New Delhi-8

1968

Price Rs. 5/00 *(Fifty only)*

PREFACE

Brahmagupta is one of the most distinguished mathematicians and astronomers of India of the Sixth Century A D He was born in 520 Śaka or 655 Vikrami or 589 A D and in 628 A D at the age of 30 he wrote his well known treatise the *Brahmasphuṭasiddhānta* and in 665 A D another small but a very significant book the *Khaṇḍakhādyaka* In 860 A D another celebrated mathematician and astronomer, [illegible] vāmi wrote his famous commentary known as the *Vāsanā* [illegible] on the *Bra* [illegible] *siddhānta* It is a matter of satisfaction that the *Indian Institute of Astronomical and Sanskrit Research* has published not only a critical edition of the *Brahmasphuṭasiddānta* but also the *Vāsanā Commentary* and a Hindi Commentary and where the Vāsanā Commentary was not available Mahāmahopādhyāya Paṇḍita Sudhākara Dvivedi s Commentary has been incorporated

Prior to Brahmagupta Āryabhaṭa (born 476 A D) wrote his celebrated treatise the *Āryabhaṭiyam* in 499 A D Āryabhaṭa was a great leader of one school of thought and his influence on the scholars of Indian astronomy was dominating Brahmagupta made certain advances on the doctrines of his predecessors , he was opposed to conservatism In fact Brahmagupta was a great critique of his times and was known for his great originality He lived in a dynamic age when the Greeks also not only penetrated into the Indian Territories their literature and thought also started gaining prominence In fact during this period India and Greece both came closer and took part in cultural impacts and advances Brahmagupta in his treatise has many a time criticised some of the Greek methods of calculations and refuted the claims of foreigners for a greater accuracy Brahmagupta s great works, the *Siddhānta* as well as the *Khaṇḍakhādyaka* took astronomy to Arabs and through them it spread to many countries in Europe

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beginning of the Kaliyuga, (ii) the diameters of the Sun Moon and the Earth and the distances of the Sun and Moon from the Earth, (iii) sidereal revolutions of the apogees of the planets in a Kalpa, (iv) sidereal revolutions of the nodes of the planets in a Kalpa (v) peripheries of the epicycles of the planets (vi) mean diameters of the planets (vii) inclination of the orbits of the planets to the ecliptic (viii) longitudes of the junction stars and (ix) celestial latitudes of the junction stars, and for such tables I am indebted to the calculations given in the *Laghubhāskarīya* and the *Mahābhāskarīya* by my esteemed friend K S Shukla who has very ably edited and translated these texts

Brahmgupta describes a number of appliances and instruments which could be used for astronomical studies A chapter has been devoted to this subject also I am sure this small Monograph on Brahmagupta and his contributions to mathematics and astronomy would be read with interest It is needless to say that in this study as in many others I have been given encouraging opportunities by Pt Ram Swarup Sharma the Director of the Indian Institute of Astronomical and Sanskrit Research and we all are indebted to the authorities of this Institute for this publication

University of Allahabad — Satya Prakash
15 November 1968

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BRAHMAGUPTA
AND
HIS WOROKS

CHAPTER I

Astronomy in Ancient Nations

Brahmagupta s great works like the *Khandakhadyaka* and *the Brahma sphuta-siddhanta* took astronomy to Arabs through whom it spread to many countries of Europe Al Beruni records this testimony in his great book on India It is doubtful that astronomy had its birth in Greece and China From remote ages China India Greece Arabia and Egypt developed the entire system in close cooperation This knowledge must have spread from their common cradle home where man for the first time developed his culture and civilisation In this chapter we propose to give a review of astronomy as developed in many of these ancient lands especially Arabia people of which land came in close contacts with India much before any recorded time

Dawn of Astronomy

The earliest man must have been the primitive astronomer The striking spectacles presented to him by the varied appearances of a sky covered with thousands of twinkling and non twinkling objects of different degrees of brightness, apparently revolving round the Earth and the daily changing phases of the Moon must have raised strange feelings of the most primitive man also. Then he must have in course of time observed the bright morning and evening stars and at a considerably late stage the comets and shooting stars and then on occasions eclipses of the Sun and the Moon These phenomena not only raised feelings of admiration but in different sections of human society often feelings of superstitious alarm By and by stars became guides for the traveller by land and sea In the midst of these observations one discovered various cycles cycle of day and night cycle of seasons and cycle of other details. Then there was a striking observation of the tides in a sea changing with the phases of the Moon

the year, the same stars are seen at corresponding hours of the night. Of course this circumstance was less conspicuous than the regular variation of the Sun's altitude in the sky as the year progresses It is the surmise that the striking naked-eye cluster, the Pleiades, must have been one of the earliest noted star-groups, and it became the first star-group for providing the first fairly close determination of the length of the year as approximately 365 days The rising of this cluster in the evening was a mark of the coming winter to primitive man; and the husbandman judged the time of reaping by its rising, and of ploughing by its setting in very ancient times, Sirius, Arcturus, the Hyades and Orion were similarly equally useful to him The passages in the *Taittiriya Samhita* and in the *Śatapatha Brahmaṇa* clearly indicate the confusion once created by following the concept of lunar months without further adjustments:

> 'Now the seasons were desirous to have a share in the sacrifice among the gods and said, 'Let us share in the sacrifice Do not exclude us from the sacrifice! Let us have a share in the sacrifice!' The gods, however, did not approve of this The gods, not approving, the seasons went to the Asuras, the malignant, spiteful enemies of the gods Those (Asuras) then throve in such a manner that they (the gods) heard of it, for even while the foremost (of the Asuras) were still ploughing and sowing those behind them were already engaged in reaping and threshing; indeed even without tilling, the plants ripened forthwith for them. (*ŚBr* I 6 1 1-3)

The Zodiac

It is difficult to say how much time it must have taken, but in fact, it was eventually noted that the Sun and Moon travel over very similar paths among the stars during their circuit of the sky This led to the formation of the Zodiac and its constellations, the centre of this zone, a belt about 16° broad, being the annual path of the Sun or Ecliptic The division into twelve parts, each corresponding to a month of the Sun's movement, was made, and their connection with the solar course during the year was found by observations of heliacal risings or settings These were the times of the year when certain bright stars would first be seen to rise before the Sun, or when

they were last seen to set after sunset In the case of Sirius the brightest fixed star, these would happen when the Sun was about ten degrees below the horizon For the less bright stars the angle would be a larger one

It must have been almost simultaneously observed that the Moon in going like the Sun round the heavens always in the same direction from west to east (i e, opposite to the diurnal motion which she shares with the other bodies) kept in general to the same track in the sky After a time, however, it must have been noted by careful observers that this path was not constant, but deviated from the centre line of the Zodiac, getting away from that line up to a maximum deviation on either side but slowly returning to it In the course of a number of years it must have become evident that the Moon's path among the stars does not lie always in the same line on the celestial sphere but in a zone or band about twenty moon breadths (10°) wide occupying the middle of the Zodiacal zone itself

Among the bright stars Mercury, Venus Mars, Jupiter and Saturn (the first two of which are never seen very far from the Sun in the sky) soon must have been noted to be moving in the Zodiac with varying periods The English name *planet* is derived from Greek *planetes* meaning a wanderer, since the planets change their positions among the Zodiacal stars

There is a word *Str* which in the *Rgveda* always occurs in the instrumental plural, *Stṛbhiḥ* The English word star is derived from this word Parāśara and Gṛtsamada I have shown elsewhere, were the first amongst the great observers, inspired by the Rgvedic hymns and Vāmadeva identified Bṛhaspati or the Jupiter planet and Vena Bhārgava discovered tht planet Venus which still bears the name of its discoverer

Constellations

Long before the Zodiacal belt was divided into 'signs' (700 B C) a number of asterisms or the configuration of stars in the sky had been arranged the brighter stars of these configurations thus identified, proved very useful in indicating the seasons of the year by the times of their rising or setting and also in locating the positions on the celestial vault of such moving objects as planets comets and shooting stars and in helping the traveller by land or sea to determine direction

These named constellations date back to very early period In India Gārgya is the name of an astronomer who is associated with a hymn of the *Atharvaveda* which for the first time enumerates constellations In many of these constellations the stars form a well marked group clearly separated from other groups and the names given to these formations are supposed to have been suggested by a resemblance to the shapes of certain familiar objects Of course the resemblance is usually very slight and depended merely on a fancy

It is remarkable that different countries devloped almost similar notions regarding these constellations The late Dr A C D Crommelin considered that there is a reason to believe that the stars may have been grouped to some extent by the Egyptians as early as 4000 B C and he remarked on their use of the then Pole Star for orienting the Creat pyramid Again Chinese are said to have mapped out the sky into many divisions of stars by 2500 B C if one can rely on their records

The idea of constellations takes us to a date much earlier than 2500 B. C even In total forty eight have come down from extremely ancient times but these do not cover the entire extent of the sky The part not occupied by any of them evidently did not rise above the horizon where the early astronomers to whom we owe their naming lived and the stars concerned were therefore not included in their constellation schemes The centre of this part (near the bright star Achernar) must have been near the South Pole of the heavens of the time and its angular radius from the Pole gives us roughly the latitude of their homes The date appears to have been about 2800 B C when owing to the precession of the equinoxes the South celestial pole was in the position indicated The latitude seems to have been about 38° North These are the findings of E W Maunder (*Astronomy without a Telescope* p 5, 1902) but from the same considerations Dr Crommelin assigns a latitude of 30° and a date 2460 B C and Proctor 2200 B C Maunder also suggested that the presence of the Lion and Bear among the stellar configurations and the absence of Elephant Tiger Camel and Crocodile seem to exclude *India towards the East and* the countries towards the West the latitude and the longitude indicated being those of Asia Minor or Armenia The suggestion that the blank area in

the sky referred to gave an approximate date for the formation of the constellations appears to have beeen first put forward in 1807 by Carl Schwartz, for some time Swedish Consul at Baku.[1]

Indo-Greek Contacts

It is highly improbable that before Alexander, there had been absolutly no contacts between India and the distant nations. Even in pre-Alexandrian era, there had been such migration is clearly evinced by the philological and mythological studies. But we do not possess historic record of it.

The conquests of Alexander the Great made the Greeks acquainted with the Eastern world, which had up to that time been visited probably by very few Europeans, and it likewise spread Greek culture to all the countries which the victorious Macedonian had been able to reach. The Indian province of his Empire became independent soon after Alxander's death and though the spread of Buddhism in the third century B. C. checked the progress of Hellenism in Northern India, the rise of the Greek kingdom of Bactria and its gradual extension south and east continued for a long time to keep alive the connection between India and the West. Not only (as has beeen asserted) the Greek and Indian drama and architecture have been strongly influenced by Hellenistic and Indian contacts, it is beyond a doubt that the entire astronomy of the two great nations is the offspring of these mutual contacts.

In earliest times astronomy had only been cultivated in India and in no other country. Some idea had been acquired during those days of the periods of the Sun and Moon and the planet Venus and Bṛhaspati (Jupiter), which were used for chronological purposes, the lunar motions being specially connected with the proper times for sacrificial acts. The Vedic era was discovered during this period by Viśvāmitra, and Gārgya enumerated the Nakṣatras. Lagadha composed his *Vedāṅga Jyotiṣa*, which is the *first* book on astronomy written in human literature. India developed her geometry in connection with the construction of sacrificial altars, and its account is found in the *Śulba Sūtras* of Baudhāyana, Āśvalāyana and Kātyāyana. Āryabhaṭa laid the foundations of algebra. One might still say that there is no sign of

1. See, Peter Doig : *A concise History of Astronomy*, London, 1950.

Earth as a Sphere

The astronomers of the *Siddhāntas* taught that the Earth is a sphere, unsupported in space and they reject the ancient mythological notion that it is supported by some animal like śesanāga (*serpent*), kacchapa (*tortoise*), or *diggajas* (*elephants*) which in turn rest on another, and so on until the support of the last one after all has to be left unexplained Bhāskara II, about A D 1150, who comments on the absurdity of this also rejects the idea that the Earth is perpetually falling since it would fall faster than an arrow shot upwards, on account of being heavier, so that an arrow could never again reach the Earth [1] Round the Earth the planets are moving, all with the same linear velocity The diameter of the Earth is 1600 yojanas the distance of the Moon is 51,570 yojanas (or 64 5 times the radius of the Earth, nearly equal to Ptolemy s greatest distance 64⅙) while the distances of the other planets result from the assumption of equal velocities [2] The equation of centre of the planets is found by an epicycle and to this arrangment the Hindus add one of their own invention, by assuming that the epicycle had a variable circumference greatest when the planet is at apogee or perigee and least at 90° from these when the equation reaches its maximum This contrivance of an oval epicycle was by some astronomers applied to all the planets, by others (Brahmagupta and Bhāskara) only to Mars and Venus by others it was altogether rejected [3] Why they complicated the calculation in this way is not clear Āryabhaṭa I of Kusumapura or Pāṭaliputra, born A D 476 made another deviation from the Alexandrian doctrines as appears in the *Brāhma sphuta siddhānta* of Brahmagupta wherein he quotes the following from Āryabhaṭa The sphere of the stars is stationary and the Earth making a revolution produces the daily rising and setting of stars and planets Brahmagupta rejects this idea saying If the Earth moves a minute in a prāna then whence and what route does it proceed? If it revolves why do not lofty objects fall? But his commentator Caturveda Pṛthūdaka Svāmi replies Āryabhaṭa s

1 *As Res* XII p 229 (*Essays* II p 394)

2. The distances are proportional to the orbital periods of revolution but for Mercury and Venus to the periods in the epicycles

3 For further details see *As Res* II p 251 (Davis) and XII p. 236 (Colebrooke also *Essays* II. p 401)

opinion appears nevertheless satisfactory since planets cannot have two motions at once and the objection that lofty things would fall is contradicted for every way the under part of the Earth is also the upper since wherever the spectator stands on the Earth s surface even that spot is the uppermost spot [1]

Earth rotation by a current of aerial fluid

It is very interesting to see the theory once advocated by Herakleides of Pontus transplanted on Indian soil especially when we remember that Seleukus, the Babylonian had adopted that theory From Babylon the theory might easily find its way to India though it is of course equally possible that Āryabhata quite independently of his Greek precursors hit on the same idea He appears to have accounted for the Earth s rotation by a wind or current of aerial fluid the extent of which according to the orbit assigned to it by him corresponds to an elevat on of little more than a hundred miles (114) from the surface of the Earth or fifteen yojana s while he put the diameter of the Earth equal to 1050 yojanas (of 7 6 miles each[2]) This was in accordance with the general opinion of the Indians that the planets are carried along their orbits by mighty winds with the same velocity and parallel to the ecliptic (while one great vortex carries all stars round the Earth in twenty four hours, but that the planets are deflected from these courses by certain invisible powers having hands and reins with which they draw the planets out of their uniform progress The power at the apogee for instance constantly attracts the planet towards itself alternately with the right and left hand (like Lachesis in Plato s *Republic*) while the deity at the node diverts the planet from the ecliptic first to one side and then to the other And lastly the deity at the conjunction causes the planet to move with variable velocity and to become occasionally stationary and even retrograde This is gravely set forth in the *Sūrya Siddhānta* and even Bhāskara gives the theory in his notes, though he omits it from his text Similarly Brahmagupta although he gives the theory of eclipses, affirms the existence of an eighth planet Rāhu which is the immediate cause of eclipses and he blames Varāhamihira

1 *Asiat Res* XII, p 227 Colebrooke s *Essays* II, p 392.
2. Colebrooke *Notes and Illustrations to the Algebra of Brahmagupta*, p xxxviii *Essays* II, p 467

Āryabhaṭa and others for rejecting this orthodox explanation of the phenomenon.[1]

Indian astronomy some times appears to be a curious mixture of old fantastic ideas and sober geometrical methods of calculation. But it is wrong to presume that these geometric calculations were derived from froeign contacts Indians have always been fond of geometry (from the earliest times of the Vedic rituals), and they from the very beginning realised the importance of applying geometry to astronomy. Side by side we find Greek contacts also. As remarked by Colebrooke, the absence of the most characteristic parts of Ptolemy's system, the equant and the details of the theories of the Moon and Mercury seems to indicate that Greek planetary theory must have been *introduced in India between the times of* Hipparchus *and Ptolemy*; and with the exception of the epicycle from the circular form, the Hindus did not modify the theory or perfect it in any way. *The precession of the equinoxes they held to consist* in a liberation within the limits of 27° (Āryabhaṭa says 24°) east and west of its mean position, but they came much nearer to the truth than Ptolemy did as regards the annual amount, as they supposed the space travelled over in a century to be $1\frac{1}{2}$°.

Contacts with Arabs

Notwithstanding some isolation of India from Europe during the Middle Ages, her astronomy was destined to exercise an indirect influence on the progress of astronomy. Through the conquest of Persia *in the seventh century*, the Arabs, like the Greeks a thousand years earlier, came in contact with India, from whence physicians and astrologers found their way to the court of the Caliph already before the reign of Harun al Rashid. We possess a detailed account of the manner in which the Indian astronomy was introduced at Baghdad, from the pen of the astronomer Ibn al Adami (who died before 920), confirmed by the celebrated memoir on India by Al Beruni, written in 1031[2]. In the year 156 of the Hijra (A. D 773), there appeared before the Caliph Al Mansur a man who had come from India; he was skilled in

1. *Asiat. Res.* XII, pp. 233, 241. *Essays*, II, pp. 398, 407.
2. Hankel, *Zur Geschichte der Mathematik im Alterthum* and *Mittelalter*. Leipzig, 1874, p. 229, Cantor, *Gesch. d. Math.* I, p. 656.

the calculus of the stars known as the Sindhind (i e Siddhānta), and possessed methods for solving equations founded on the kardagas (i e kramajyā, sines) calculated for every half degree, also methods for computing eclipses and other things Al Mansur ordered the book in which all this was contained to be translated into Arabic, and that a work should be prepared from it which might serve as a foundation for computing the motions of the planets This was accordingly done by Muhammed ben Ibrahim Al Fazari whose works the Arabs call the great *Sindhind* and from it an abstract was afterwards made for Al Mamun by Abu Giafar Muhammed ibn Musa al Kwarizmi, who made use of it to prepare his tables which obtained great renown in the lands of Islam But when Al Mamun became Caliph he promoted these noble studies and called in the most learned men in order to examine the *Almagest* and make instruments for new observations

Arabs and Greeks

The account of which the above is an abstract shows us clearly the origin of the study of astronomy and mathematics under the Abbasid Caliphs But though the first impulse came from India the further development of Arabian science was to a considerable extent founded on that of Greece and Alexandria It was through the court physicians from the flourishing medical school kept up by Nestorian Christians of Khusistan that a knowledge of Greek Philosophy and science was first spread among the subjects of the Caliphs and by degrees the works of Aristotle Archimedes Euclid Apollonius, Ptolemy, and other mathematicians were translated into Arabic. Fresh translations of Ptolemy were made from time to time in the various kingdoms into which the vast empire of the Caliph was soon split up,[1] and a thorough knowledge of Ptolemaic astronomy was thus spread from the Indus to the Ebro There were several special inducements for Muhamedans to pay attention to astronomy, such as the necessity of determining the direction in which the

1. The earliest is probably that of Al Hajjag ben Jusuf ben Matar early in the ninth century See Suter *Die Mathematiker und Astronomen der Araber und ihre Werke* Leipzig 1900 (p 9) which valuable bibliographical summary has been follwed by J L. E. Dreyer as regards names and dates (J L. E. Dreyer's *A History of Astronomy* 1953 we have reproduced this account from his chapter XI.)

faithful had to turn during prayers, also the importance of the lunar motions for the calendar, and the respect in which judicial astrology was held all over the East. The Caliph Al Mamun, son of Harun Al Rashid (813-833) is the *first* great patron of science, although the Omayyad Caliphs had much earlier an observatory near Damascus, and the Jew Mashallah (who died about 815) had already before the reign of Al Mamun won a name as an observer and astrologer. But the Damascus observatory became quite eclipsed by that erected at Baghdad in 829 where continuous observations were made and tables of the planetary motions constructed while an important attempt was made to determine the size of the Earth. Among the astronomers of Al Mamun and his successors one of the greatest was Ahmed ben Muhammed Al Fargani (afterwards known in the West as Alfraganus), whose *Elements of Astronomy* were translated into Latin in the twelfth century and contributed greatly to the revival of science in Europe.[1] Tabit ben Korra (826-901) was a most prolific writer and translator, but is chiefly known in the history of astronomy as a supporter of the erroneous idea of the oscillatory motion of the equinoxes. A younger contemporary of his, Muhammed Al Battani (died 929), was the most renowned of all the Arabian astronomers and became known in the West in the twelfth century (under the name of Albategnius) by the translation of the introduction to his tables.[2] Already in his time the power of the Caliphs had commenced to decline, and they soon lost all temporal power. The study of astronomy was, however, not influenced by this loss of patronage, as the Persian family of the Buyids, who in 946 obtained possession of the post of Amir-al-Omara (corresponding to the Frankish Major Domus) took over the *role* of patrons of science, so long and so honourably carried on by the Abbasid Caliphs Sharaf al Daula built in 988 a new observatory in the garden of his palace, and among the astronomers who worked there was Muhammed Abu 'l Wefa al Buzjani (959-998), who wrote an *Almagest* in order to

1. First printed at Ferrara in 1493. See the edition of Golius, Amsterdam, 1669.

2. Translated by Plato of Tivoli. First Printed in 1537 after the book of Alfargani. Dreyer has used the edition of Bologna, 1645, and a new edition which is now being published by C. A. Nallino, of which the Arabic and a Latin translation of the text have already appeared (*Pubbl. d. R. Osservator o di Brera in Milano*. No. 40, 1899-1903).

as he owed all he knew about the science to the example and the teaching of Muhammedans and Jews King Alfonso X. of Castille named el Sabio (1252 1284), followed the example of the Caliphs and called astronomers to his court to assist in the preparation of the renowned *Alfonsine Tables*

With Alfonso the study of astronomy disappeared from Spain, but not before it had been revived in the East In 1258 the still existing but shadowy Caliphate of Baghdad was swept away by the Mongol conqueror Hulagu Khan, grandson of Genghis Khan but already in the following year this great warrior listened to the advice of his new vazier, Nasir ed-din al Tūsi (born at Tūs in Khorasan in 1201, died in 1274), and founded a great and magnificent observatory at Merāgha, in the north-west of Persia. In this observatory, which was furnished with a large number of instruments, partly of novel construction, Nasir ed-din and his assistants observed the planets diligently and produced after twelve years labour, the "Ilokhanic Tables." Among the astronomers of Merāgha seems to have been Juhanna Abu 'l Faraj, called Bar Hebrāyā or the son of a Jew He was a Christian, born in 1226 and from 1264, till his death in 1286 Maphrian or Primate of the Eastern Jacobites He left a well-known chronicle and an astronomical work, both written in Syriac, as well as other writings [1] The observatory at Merāgha had not a long life, and Asiatic astronomy had to wait a century and a half, until the grandson of another terrible conqueror erected another observatory Ulug Begh, grandson of Tamerlan, drew learned men to Samarkand and built an observatory there about the year 1420 where new planetary tables and a new star catalogue the first since Ptolemy's, were prepared Ulug Begh died in 1449, he was the last great Asiatic protector of astronomy, but just as the Eastern countries saw the star of Urania setting it was rising again for Europe

In this review of Arabian astronomers we have only mentioned a few, omitting several names of distinction, whose

1 *Le livre de l ascension de l esprit sur la forme du ciel et de la terre* Cours d' Astronomie redige en 1279 par Gregoire Aboulfarag dit Bar Hebraeus Publie par F Nau Paris 1899-1900 (2 parts Syriac and French) His chronicle is the chief authority for the fable about the burning of the Alexandrian Library by order of the Caliph Omar For a very thorough refutation of this see Butler, *The Arab Conquest of Egypt* Oxford, 1902 pp 401-426.

owners devoted themselves to other branches of astronomy Though Europe owes a debt of gratitude to the Arabs for keeping alive the flame of science for many centuries and for taking observations some of which are still of value it cannot be denied that they left astronomy pretty much as they found it They determined several important constants anew but they did not make a single improvement in the planetary theories It will therefore be sufficient to enumerate the improvements attempted and the opinions held by Arabian astronomers with out keeping strictly to the chronological order although we are here dealing with a period of about six hundred years and m n belonging to very different nations who had little in common except their religion and the language in which they wrote

Figure of Earth

Turning first to the question of the figure of the Earth we find a remarkable contrast between Europe and Asia In the world under Islam there was an entire absence of that hostility to science which distinguished Europe during the first half of the Middle Ages Though we learn from Kazwini s *Cosmo graphy*[1] that some of the earlier Arabs believed the Earth to be shaped like a shield or a drum still there is no record of any Arabian having been persecuted for asserting that the Earth is a sphere capable of being inhabited all over Whether this was in consequence of the warriors of the Caliphs having carried their arms to the centre of France on one side and to the borders of China on the other while their merchants travelled south ward to Mazambique and northward to the centre of Asia is another question anyhow the fact of the Earth being a sphere of very small dimensions in comparison to the size of the uni verse was accepted without opposition by every Arabian scholar and the very first scientific work undertaken after the rise of astronomy among them was a determination of the size of the Earth It was carried out by order of the Caliph Al Mamun in the plain of *Palmyra* According to the account given by Ibn Junis the length of a degree was measured by two observers between Wamia and Tadmor and by two others in another loca lity we are not told where The first measure gave a degree

1 *Zakarija Ben Muhammed Ben Mahmud El Kazwini s Kosmographie* deutsch von H Ethe Leipzig 1868 p 295.

equal to 57, the second one equal to 56¼ Arabian miles of 4000 black cubits, and *the approximate mean*, 56⅔ *miles*, was *adopted* as the final result, the circumference of the Earth being 20,400 miles and the diameter 6500 miles. Another report, by Ahmed ben Abdallah, called *Habash*, an astronomer under Al Mamun (quoted by Ibn Jūnis), states that a party of observers (no names given) proceeded along the plain of Sinjar until they found a difference in meridian altitudes, measured the same day, equal to one degree, while *the distance travelled over was found to* be 56¼ miles[1]. Probably two different determinations were made. If the "black cubit" is the Egyptian and Babylonian cubit of 525 mm.[2], the mile *would* be=*2100* m. and 56⅔ *miles*=*119,000* meters, rather a large result.

The doctrine of the spherical earth remained undisputed in the Muhammedan learned world, though the curious error of assuming that the level of the sea was higher on some parts of the Earth than on others appears to have found some adherents among Arabian writers as well as in Europe[3] We may, therefore, at once pass on to the motions of the heavenly bodies. Al Battani determined the longitude of the Sun's apogee and found it=82° or 16° 47' more than Ptolemy had given. As he believed

1. Caussin, *Not. et Extraits*, vii pp 94-96; Delambre, *Hist. de l'astr du Moyen Age*, pp 78 and 97, Shems ed-din, *Manuel de la cosmographie*, traduit par Mehren, Copenhague, 1874 p. 6. Suter, p 209, mentions a third report (from Ibn Challikān's *Biographical Dictionary*), according to which the sons of Musa first measured in the plain of Sinjar and afterwards as a test at Kūfa, by order of Al Mamun The eldest of the sons of Musa died 41 years after Al Mamun, and the names of the observers in the first report are different, so that the third report is not to be relied on Al Fargani merely gives 56⅔ miles as the result of Al Mamun. According to Shah Cholgii *Astronomica . . studio et opera Ioh. Gravii* London, 1652, p 95, Ala ed-din Al Kusgi (one of the Ulug Begh's astronomers) gives the circumference of the earth =8000 parasangs As a persian parasang =30 *stadia* (Hultsch, *Griech u Rom Metrologie*, p. 476) this would seem to be the value of Posidonius, 240,000 stadia Kazwini (p. 298) gives the circumference =6800 parasangs on the authority of Al Beruni.
2. Hultsch p. 390
3 It deserves to be mentioned that Shems ed-din of Damascus (1256-1327) explains the great Preponderance of dry land in the northern hemisphere by the attraction of the Sun on the water, which is the greatest when the Sun is in perigee, at which time it is nearly at its greatest south declination. That this accumulation of water would not be a permanent one does not occur to him (*Cosmographie*, p. 4).

that Ptolemy's value had been found by himself,[1] and as he adopted 54″ (or 1° in 66 years) as the annual amount of precession, there remained (assuming that 760 years had passed since the time of Ptolemy) an outstanding error of 79″—54″=25″ per annum In reality the annual motion of the solar apsides is 11¼″, still we may say that the discovery of this motion is due to Al Battani, though he did not announce it as such, in fact he merely gives his own value as an improvement on that of Ptolemy Even Ibn Junis (who found 86° 10′) did not suspect that the apogee was steadily moving but merely says that it must be corrected for precession (1° in 70 years), and remarks that the longitude of the apogee is very difficult to determine accurately[2] On the other hand, Al Zarkali found a smaller value, 77° 50′ and as he also found a smaller value of the eccentricity he thought it necessary to let the centre of the Sun's eccentric orbit describe a smaller circle, after the example set by Ptolemy in the case of Mercury[3] The inclination of the ecliptic which the Greeks had found—23° 51 20″ was by the astronomers of Al Mamun found—23° 33′ (in 830), by Al Battani (in 879) and by Ibn Junis 23° 35′[4] When Al Zarkali found 23° 33, he, and afterwards Abu 'l Hassan Ali of Morocco concluded that the obliquity oscillated between 23° 53 and 23° 33′, an idea to which the prevailing belief in the "trepidation" of the equinoxes lent countenance[5]

Moon and its orbit

If we now turn to the Moon we do not find that the Arabs made any advance on Ptolemy Several of them noticed that the inclination of the lunar orbit was not exactly 5°, as stated by Hipparchus Thus, Abu 'l Hassan Ali ben Amagiur early in the tenth century says that he had often measured the greatest latitude of the Moon and found results greater than that

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1 *Scient Stell* Cap xxviii Bologna 1645 p 72, Nallino, p 44 At the end of Cap xlv he says the apogees of the Sun and Venus are both in 82° 14′ and Ibn Junis also gives 82° 14′ as the value found by Al Battani (Caussin, p 154)

2 Caussin pp 232 and 238. Abu l Faraj gives 89° 28′ for the year 1279 (p 22)

3 Sedillot *Prolegomenes aux tables astron. d' Oloug Beg* (1847) pp lxxx lxxxii Riccioli, *Almag Novum* I p. 157

4 Caussin, p 56. For A D 900 Newcomb gives 23° 34′ 54″ with a diminution of 46″ per century so that the Arabian astronomers erred less than 1′

5 Aboul Hassan Ali *Traite des Instruments astron des Arabes* T.I.p 175; Sedillot *Memoire sur les instr astr des Arabes*, p 32

of Hipparchus, but varying considerably and irregularly. Ibn Junis, who quotes this, adds that he has himself found 5° 3′ or 5° 8′, while other observers are said to have found from 4° 58′ to 4° 45′.[1] Want of perseverance and of accurate instruments caused them to miss a remarkable discovery, that of the variation of the lunar inclination.

Abu 'l Wefa and his Almagest

But an even more remarkable discovery has been claimed for an Arabian astronomer. In 1836 the younger Sedillot announced that he had found the third inequality, the variation, distinctly announced in Abu 'l Wefa's *Almagest*. A fierce controversy raged for a number of years as to the reality of this discovery, Sedillot alone defending his hero with desperate energy and refusing to listen to any arguments, while Biot, Libri and others as strenuously maintained that Abu 'l Wefa simply spoke of the second part of the evection, the prosneusis of Ptolemy. The fight had died out when, in 1892, Chasles *suddenly took up the cudgels for Sedillot and pointed out what seemed to him to be some contradictions in Ptolemy's statement.*[2] Nobody answered this until Bertrand did so in 1871; he called attention to several inaccuracies in the text of Abu 'l Wefa as we possess it now, and also showed that Abu 'l Wefa did not add his "mohazat" to the prosneusis, the latter not being included in his "second anomaly."[3] It is unnecessary to enter into a more detailed account of the controversy; but to show that any weapon was considered good enough with which to defend Abu 'l Wefa, it may be mentioned that Sedillot and Chasles tried to prove that Tycho Brahe must have copied his discovery from Abu 'l Wefa, because he calls it *hypothesis redintegrata* Tycho used this same phrase in speaking of his own planetary system, which he most emphatically claimed as

1. Sedillot, *Prolegomenes*, p. xxxviii. *Materiaux pour servir a l' hist. des sciences chez les Grecs et les Orientaux*, T. I, p 283. The sons of Musa ben Sakir (about 850) seem to have been the first to find a value differing from that of the ancients Abraham ben Chija, a Jewish writer who lived about A. D. 1100 says that Ptolemy found 5°, but that according to the opinion of the Ishmaelites it is 4½° (*Sphaera mundi*, Basle 1546 p 102)
2. *Lettre a M Sedillot sur la question de la variation lunaire*. Paris, 1862, 15 pp 4° and *Comptes Rendus*, vol. 54 p 1002.
3. *Comptes Rendus* vol 73, pp 561, 756, 889, *Journal des Savants*, 11 Oct. 1871.

facts, as Biot has shown *from the Ptolemy's numerical data that* the deviation of the line of apsides reaches its maximum value of $\pm$ 13° 8′.9 in elongations 90° $\mp$ 32° 57′.5.[1] But it must be acknowledged that the words in question are also used very vaguely, e. g. by Abu 'l Wefa himself, who says that the velocity of the superior planets after emerging from the Sun's rays diminishes gradually till their distance from the Sun is about a *tathlith*, when they become stationary. It looks almost as if these words might be used to denote any elongation outside syzygy and quadrature.[2]

If Abu 'l Wefa had made a new discovery, we should have expected later Arabian astronomers to have alluded to it. But not one of them gives anything but interpretations of the lunar theory of Ptolemy, and in expressions very similar to those employed by Abu 'l Wefa. Attention was at once called to this fact, and Isaac Israeli of Toledo (about 1310) and Geber of Seville were quoted as examples,[3] though it would, of course, have been quite possible for these two writers to have remained ignorant of whatever progress astronomy might have made in the school of Baghdad. But this objection does not apply to Nasir ed-din al Tusi, in whose review of the Almagest and Memorial of Astronomy the inequalities known to Ptolemy and no others, are described and credited to Ptolemy[4]; not to Mahamud al Jagmini (about 1300), who wrote a compendium (*mulachchas*)

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1 *Journal des Savants*, 1843, p 701 ("Sur un traite arabe relatif a l' astronomie," Reprint, p. 47) This deviation does not represent the amount of the correction to the Moon's place as seen from the Earth, so that there is not any contradiction in Ptolemy's account.

2. Carra de Vaux l. c p. 466 The Arabs had no word for "octants." Nasir-ed-din on one occasion wants to mention them, and has to call them "*the points midway between syzygy and quadrature.*"

3 Isaac Israeli repeatedly speaks of these inequalities discovered by Ptolemy, two of which are not found at conjunction and opposition *Liber Jesod, Olam seu Fundamentum Mundi auctore R. Isaac Israeli Hispano*, section III ch. 6 and sect v. ch 16, Part I p. xxiv Part II. p xxxi (*Berlin*, 1848 and 1846 ; this publication is not mentioned by Carra de Vaux)

4 C. de Vaux, "Les spheres celestes selon Nasir Eddin Attusi", Appendix to P. Tannery's *Recherches sur l' astr. anc* p 342, and *Journ asiat.* 1892, p. 459 · "The third anomaly is that of the prosneusis ; it is called the equation of the proper motion" (i e of the motion on the epicycle).

of astronomy [1] Nor can any objection be raised to Abu l Faraj (Bar Hebraeus) and it would be impossible to explain more clearly than he does the effect of the prosneusis He says The third inequality is the angle formed at the centre of the epicycle by two lines which are drawn one from the centre of the universe and the other from the point called the prosneusis at the end of which is the apogee of the epicycle at which commences the proper motion and which is called the mean apogee The apogee which is at the end of the line drawn from the centre of the universe is called the apparent one The point proseneusis is on the side of the perigee of the eccentric 10 parts 17 minutes from the centre of the world[2] which is itself at the same distance from the centre of eccentric. The *maximum value of this angle is 13 parts 9 minutes* when the Moon is a crescent or ¾ gibbous, that is near the hexagon or trigon with the Sun In fact when the epicycle is four or eight signs distant from the apogee of the eccentric, the Sun is itself two or four signs distant from [the centre of the epicycle] because it is half way between this centre and the apogee In the tables this inequality of the two apogees is called the first angle and is included in the motion of the centre [3] While this describes the construction of Ptolemy as clrealy as possible, at the same time the agreement of the account with that of Abu l Wefa is perfect Abu l Faraj even (like Nasir ed din) describes as a fourth inequality in longitude that caused by the motion along an orbit inclined to the ecliptic so that he would not have neglected to describe the variation if it had been found by an astronomer of Baghdad We may add that the Jewish writer Abraham ben Chija (A D 1100) in his *Sphaera Mundi* also describes the aberration of the apside of the epicycle chiefly in sexta et tertia parte mensis [4]

1 Translated by Rudloff and Hochheim *Zeitschrift der Deutsche Morgenland Ges* XLVII pp 213—275 He describes (p 249) how the line of apsides is directed to a point called the corresponding point and gives its position correctly The inequality he calls the deviation.

2. Nasir ed din gives 10p 9'

3 *Le livre de l ascension* &c T II PP 29-30. Two codices add after the word prosneusis This is the point mohazat.

4 *Sphaera Mundi* (1546, ed Schreckenfuchs) p 75 Munster s commentary to the Hebrew text (p 116) has cum centrum est in sextil aut trino aspect [id est quando abest a sole duobus signis aut quatuor] the words in brackets are not in the Hebrew original The words sixth and third are unmistakable (shishith and shelishith) Apparently no one has hitherto thought of consulting Abraham ben Chija.

Abu 'l Wefa and Ptolemy

Therefore, Abu l Wefa did not know a single thing about the motion of the Moon which he had not borrowed from Ptolemy But the prosneusis of Ptolemy is not the variation discovered by Tycho Brahe The latter depends solely on the elongation of the Moon from the Sun, as it is $= +39' 5 \sin 2\epsilon$, while it is beyond the power of mortal man to express the effect of the prosneusis without the anomaly Ptolemy s exprssion for all the inequalities in longitude assumed by him when developed analytically, found to contain, in addition to terms representing the equation of the centre and the evection the latter being

$$+1^\circ 19' 5 \sin \quad (2\epsilon - m),$$

a very considerable term

$$+17\ 8 \sin 2\epsilon\ [\cos (2\epsilon + m) + 2 \cos (2\epsilon - m)]$$

where ϵ is the elongation and m the mean anomaly [1]

Obviously this term has nothing in common with the variation, except that it disappears in the syzygies and quadratures Tycho Brahe did not hang his new term on to the unaltered lunar theory of Ptolemy, and by doing that we should in fact only spoil the latter and make its maximum error rise to more than a degree [2] Owing to the insufficiency of the observations at his disposal Ptolemy could only perceive that there was some outstanding inequality after allowing for the evection only appearing outside the syzygies and quadratures but he was neither able to find the law which governed the phenomenon nor was he aware what a large quantity it represented, he could only tinker up his constructions a little and in this he was most faithfully followed by the Arabs, who added nothing to what he had done and left it to the reviver of practical astronomy to discover the third lunar inequality

Al Fargani and others on Planets

Passing to the five planets we find that, generally speaking, very few attempts were made to improve the work of Ptolemy But the Arabs were not content to consider the Ptolemaic system

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1 P Tannery *Recherches* p 213 Another expans on of Ptolemy s lunar inequalities in a series was g ven by B ot *Journal des Savan s* 1843 p 703 (Reprint p 49)

2 P Kempf, *Unters chungen uber d e Ptolemaische Theorie der Mondbewegung* Berl n 1878 (Inaug Diss) p 37

merely as a geometrical aid to computation; they required a real and physically true system of the world, and had therefore to assume solid crystal spheres after the manner of Aristotle. Above the Moon is the Alacir, the fifth essence, which is devoid of lightness and heaviness, and is not perceptible to the human senses, of this substance the spheres and planets are formed [1] Already in the book of Al Fargani we find the principle adopted which we have seen dates from the fifth century (Proklus) and which became universally accepted in the Middle Ages, that the greatest distance of a planet is equal to the smallest of the planet immediately above it, so that there are no empty spaces between the spheres [2] The semidiameter of the Earth is by Al Fargani given as 3250 miles which corresponds very nearly to Al Mamun's $56\frac{2}{3}$ miles to a degree if we put $\pi = \frac{22}{7}$ Starting from Ptolemy's distances of

Greatest Distance of	Al Fargani	Al Battani	Abu 'l Faraj[3]
Moon	$64\frac{1}{6}$	$64\frac{1}{6}$	$64\frac{1}{6}$
Mercury	167	166	174
Venus	1 120	1 070	1 160
Sun	1 220	1,146[4]	1 260
Mars	8 876	8.022	8 820
Jupiter	14 405	12 924[5]	14.259
Saturn	20 110	18 094	19 963

the Moon and the Sun it was easy to express the other distances in semidiameters of the Earth, the ratios between the greatest and

1 Al Battani, cap. 50 (p 195)

2 Al Fargani cap 21 (ed Golius, p 80) Much later Maurolycus in his *Cosmographia* (Venice 1543 f, 20a) proves that Mercury and Venus must *be below the Sun by pointing out that there would otherwise be a large vacant space between the Sun and the Moon*

3. pp 189 191

4 So in Nallino s *ed* (Milan 1903 p 121) the ed of 1645 has 1176.

5 The ed of 1645 has 12 470 obviously an error as the ratio of greatest to smallest distance is given as 37.23 for Saturn 7 5 (misprinted 7 2) or quantitas unius et duarum quintarum ad unum (p 199) Nallino s ed (Milan 1903) has 12,924 Abraham ben Chija has 12,400.

smallest distances being in substantial agreement with the theory of Ptolemy Al Battani also gives a similar set of figures, though with some slight differences He does not mention peculiar treatment given by Ptolemy to the theory of Mercury. The above table gives the distance expressed in semidiameters of the Earth

Al Kusgi and diameters of planets

Al Kusgi, one of the astronomers of Ulug Begh, gives a list of the semidiameters of the "concavities" of the planetary spheres (i e the smallest distances of the spheres) expressed in parasangs, the diameter of the Earth being 2,545 parasangs[1] Expressed in semidiameters of the Earth, the figures turn out somewhat different from those given above, e g the smallest distance of the Sun being 1,452 and the greatest of Saturn 26,332, but he does not supply any means of making out how these figures were found

Before leaving this subject, we shall also give the diameters of the planets according to Al Fargani, as they became known in Europe at an early date and were quoted by Roger Bacon and others[2] *With trifling variations the same values are given by* Al Battani, Abu 'l Faraj, and Abraham ben Chija.

	Apparent Diameter	True Diameter (Earth's=1)
Moon in apogee	$31\frac{1}{3}'$ ...	1 : $3\frac{2}{5}$
Mercury, mean dist	$\frac{1}{15}$ of Sun's .	$\frac{1}{28}$
Venus " "	$\frac{1}{10}$ " ...	1 : $3\frac{1}{4}$
Sun " "	$31\frac{1}{3}'$ . .	$5\frac{1}{2}$
Mars " "	$\frac{1}{20}$ of Sun's ...	$1\frac{1}{6}$
Jupiter " "	$\frac{1}{12}$ of Sun's ...	$4\frac{1}{2} + \frac{1}{16}$
Saturn " , ...	$\frac{1}{18}$ " ...	$4\frac{1}{2}$

Al Kazwini, Abu'l Faraj and Al-Jagmini on Excentric Spheres of the Sun

The system of the spheres is set forth in greatest detail in three treatises of later date, the cosmograpy of Zakarija ben Muhammed ben Mahmud al Kazwini (about 1275), the astronomy

1 *Astronomica Shah Cholgii* pp 95-97

2. There are some slight differences between the figures given in the various editions (J.L.E. Dreyer has compared those of 1493 1546 and 1669) but those given above agree with the cubic contents according to Al Fargani. The figures of Kazwini seem to have been greatly corrupted

of Abu 'l Faraj, written in 1279, and that of Mahmud ibn Muhammed ibn Omar al Jagmini whose date and nationality are equally uncertain, but who probably wrote in the thirteenth or fourteenth century We find in these text books an elaborate system of spheres designed to account for every particular of planetary motion, in perfect agreement with each other as to the general arrangement of the spheres and offering nothing new as to lunar or planetary theory The accompanying figures (taken from Jagmini) will illustrate the ideas better than a lengthy description [1] The Sun is a solid sperical body, fitting between two excentric spherical surfaces which touch two other surfaces, in the

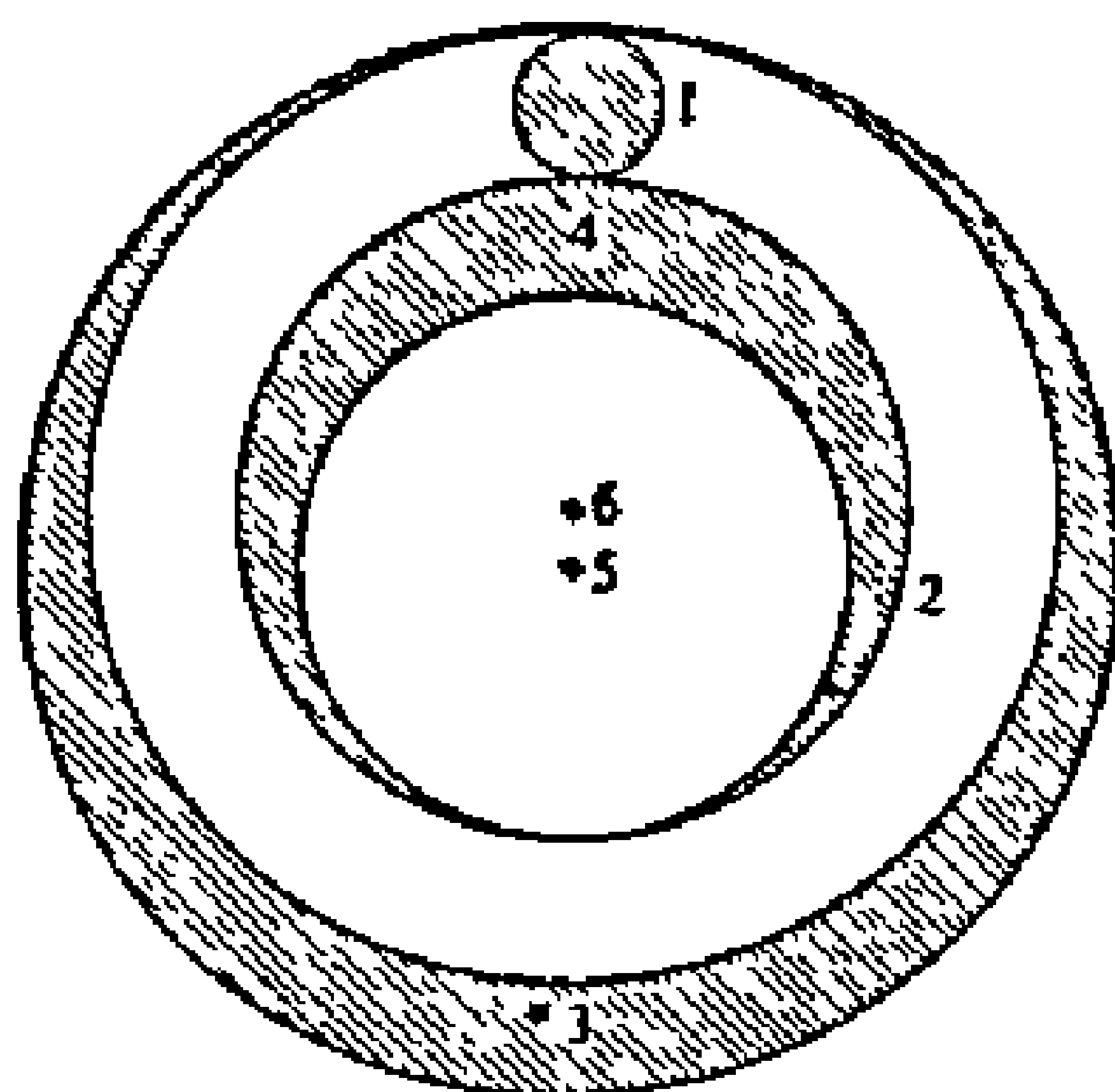


Fig 1 —Planetary motions and system of spheres

1 The Sun 2 Excentric sphere 3 The surrounding spheres 4 The complement of the surrounding sphere 5 Centre of the world 6 Centre of the excentric sphere

common centre of which the Earth is situated and which between them enclose a space (or intersphere as Abu l Faraj calls it), named by Jagmini al mumaṭṭal or the equably turning sphere which has the same motion from west to east as the fixed stars, i e precession The spheres of the three outer planets and Venus are arranged on the same plan except that the place of the body of the Sun is taken by the epicycle-sphere of each planet, to the inner surface of which the planet (a solid spherical body) is attached or (as Abu l Faraj says[2]) fixed like a pearl on a ring touching the

1 Al Kusgi gives very similar diagrams of the spheres of the Saturn, Mercury and the Moon.

2 Precession is supposed to be included in this, the first motion. The second

(Continued on next page)

surface in one point." The axis of the excentric sphere is incli ned to that of the mumattal sphere, which causes the motion in latitude The lunar system comprises an additional sphere outside the others the centre of which coincides with the centre of the world, and which is called *al gauzahar*, signifying the cosntella

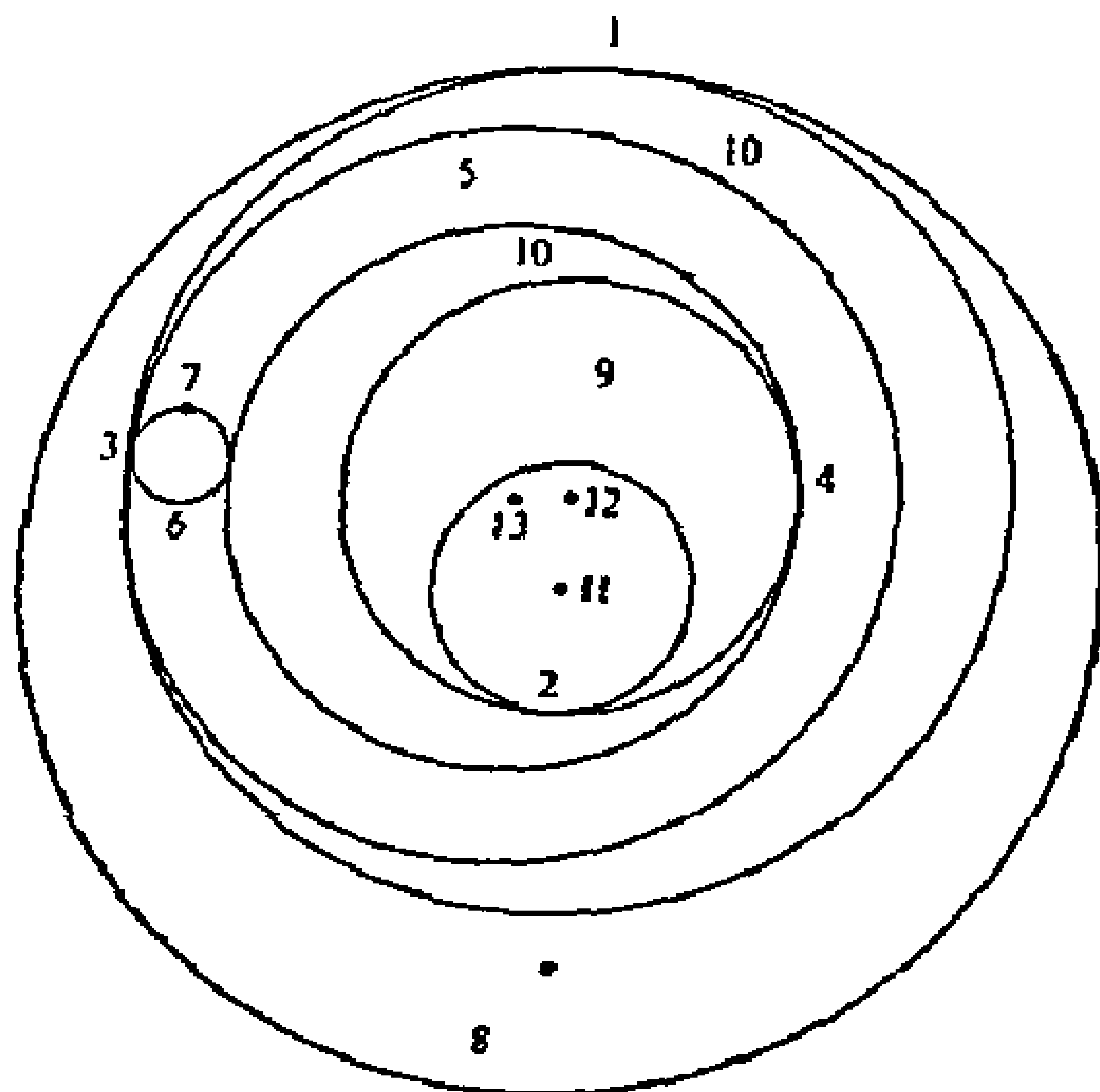


Fig 2—Spheres of Mercury

1 Upper Apsis 2. Lower Apsis Upper Apsis of deferent sphere 5 Deferent sphere 4 Lower Apsis of deferent sphere 6 Epicycle 7 Mercury 8. Surrounding complement 9 Surrounded part of Mumattal sphere 10. Mudir sphere. 11 Centre of the world. 12. Centre of Mudir 13 Centre of deferent sphere.

tion Draco, as this sphere provides for the revolution of the lunar nodes ("the head and tail of the dragon) round the zodiac. The inner one of the two concentric spherical surfaces between which the excentric sphere lies, surrounds immediately the fire sphere of the Earth The system of Mercury is more complicated, as a space had to be provided for the revolution of the centre of the

(Continued from previous page)

one is that of the con entric oblique intersphere (called the mail sphere or the sphaera deflectens) round the centre of the world 11°9' per day by which amount the lunar apogee moves towards the west. The third motion is that of the excentri carrying the centre of the epicycle 24°22' towards the east The fourth is the motion on the epicycle Abu 'l Faraj p 27.

excentric sphere The figure shows the excentric sphere enclosed in a sphere, *al mūdir* or the turning one, which allows the upper apsis or apogee of the excentric or deferent sphere (3 in the figure) to move right round the outer surface of the *mūdir* The inner surface of the mumattal sphere immediately surrounds the *gauzahar* sphere of the Moon

It was a necessary consequence of the large solar parallax of 3' accepted by Ptolemy, that Mercury and Venus must be very near the Earth, since they are assumed to be nearer than the Sun. Thus Abraham ben Chija says that the shadow or the Earth extends beyond the orbit of Mercury but does not reach that of Venus[1] Ptolemy never mentions the parallaxes of Mercury and Venus, as to which nothing was known, though they ought, of course, to be greater than 3 But on the assumption that the smallest distance of Mercury is equal to the distance of the Moon at apogee, the parallax of Mercury ought to rise to 54, which must have been felt to be too large a quantity, though it does not seem to have struck Al Battani as anything surprising, perhaps because Mercury cannot be seen when in inferior conjunction It may have been this necessarily large parallax of Mercury, which induced Ibn Jūnis (without any explanation) to reduce the solar parallax from 3 to 2', or rather to 1' 57"[2] Geber[3] blames Ptolemy for having said that the parallaxes of the planets are insensible, and remarks that he ought therefore logically to have placed Venus and Mercury above the Sun He takes great pains to show that Venus may be exactly on the line joining the Sun and the Earth Indeed, Geber neglects no opportunity of criticising Ptolemy's methods of finding the elements of the orbits,[4] and he is generally very unjust to him but he does not venture to

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1 *Sphaera mundi* ed Osw Schreckenfuchs Basle 1546 pp 84, 86.

2 Unpublished chapters of Ibn Junis, reviewed by Delambre *Hist de l'astr du Moyen Age* p 101

3 *Instrumentum primi mobilis a P Apiano Accedunt iis Gebri filii Affla Hispalensis libri IX de astronomia*, Norimbergæ 1534, fol. (Introd p 3 and lib VII, p 104).

4 See the long indictment on pp 2-3 of his introduction. He blames Ptolemy among other things for assuming that the centre of the deferent is half way between the centres of the zodiac and of the equant, while he himself deduces this from the movements.

substitute any other system and does not object to the general principles of the Ptolemaic system.[1]

Three great names :
Ibn Badja, Ibn Tofeil (Abubacer) and Abu Welid (Averroes)

Geber's attempts to pick holes in the work of Ptolemy were, perhaps, not unconnected with the rapid rise of Aristotelean philosophy in Spain in the twelfth century, which, though not destined to last long, nevertheless exercised a considerable influence on the spread of knowledge of Aristotle in the Christian world, while it cast a halo round the Caliphate of Cordova, which at that time, under the enlightened rule of the Almohades, seemed to have reestablished the glory of the best days of the Moslem world Three names are specially associated with this movement

(i) Abu Bekr Muhammed Ibn Jahya al Sayeg, called Ibn Badja (of Saragossa, died 1139), known as Avempace among the Scholastics,

(ii) his pupil Muhammed ben Abdelmelik Ibn Tofeil (of Granada, died 1185-1186) called Abubacer by the Scholastics,

(iii) and finally the greatest philosopher of Islam, Ibn Rosd Abu Welid, known as Averroes (1126-1198)

In studying Aristotle they laid special stress on his scientific works, and did not, like their Christian successors, think of little but dialectics The acceptance of the system of homocentric spheres or some modification of it must, therefore, have seemed a necessity to the Arabian philosophers and this, of course, led them to reject the theory of epicycles The little we know of the opinions of Ibn Badja on this subject is found in the famous work *The Guide of the Perplexed* of the great Jewish scholar Moses ben Maimun of Cordova, better known as Maimonides, who tells us that he had his information from a pupil of Ibn Badja Like Geber (with whose son he had been familiar), Maimonides doubted that Mercury and Venus were nearer than

1 Copernicus possessed a copy of Geber's book which is now in the University library at Upsala. On the title-page after the author's name he has written "Egregi Calumiatoris Ptolemaei" while a number of marginal notes show that he has read the book carefully Curtze, *Mittheilungen des Coppernicus Vereins* I, p 37.

the Sun, though he would not venture to say how they actually moved.[1] But what is more important, he declared the motion of a planet on an epicycle to be contrary to physical principles, because there are only three motions possible in this world: around its centre, or towards it, or away from it, while he also maintained that according to Aristotle circular motion can only take place round a real central body.[2] Though Aristotle in reality did not object to epicyclic motion with a mathematical point as centre, for the simple reason that it had not been proposed when he wrote, while as we have seen, his moving principle had nothing to do with the centre of motion, it is easy to see that Ibn Badja's real difficulty was the same which afterwards produced so many obstacles to the advance of science in Europe: whatever could not be found in Aristotle's book must be unworthy of notice. According to Maimonides (who, however, makes the reservation that he had not heard it from disciples) Ibn Badja constructed a system of his own, in which he only admitted excentric circles but no epicycles. We are not given any particulars as to this system but there can hardly be any doubt that its author confined himself to generalities and did not attempt to represent phenomena like *the lunar inequalities by it. Maimonides remarks that there is* nothing gained by Ibn Badja's reform, since the excentric hypothesis is as objectionable as the epicyclic one, as it also supposes motion round an imaginary point outside the centre of the Earth. The centre of the excentric, on which the Sun is supposed to move, is outside the convexity of the lunar sphere and inside the concavity of that of Saturn's excentric is between the spheres of Mars and Jupiter. He adds that the revolution of a number of concentric spheres around a common axis is conceivable, but not the revolution round different axes inclined to each other, as the spheres would disturb each other unless there are other spherical bodies between them. This attempt to revive and modify the system of (movable ?) excentrics did, therefore, not mend matters.[3]

1 *Rabbi Mosis Majemonidis Liber Doctor Perplexorum.* Basileae, 1629 Pars II, cap IX.

2 Ibid., Pars II, cap. XXIV

3 Maimonides also remarks (in the same chapter) that the supposed inclinations of Mercury and Venus in the *Ptolemaic* system are difficult or impossible to comprehend or imagine as really existing. Therefore if what

(Continued on next page)

(the motion of the pole of the orbit being added to or subtracted from the motion of the planet), so that the epicycle is hereby rendered superfluous. The lengths of the radii of these small circles are not given, except in the case of Saturn, where the radius is 3° 3',[1] while the mean pole of the moon is 5° (the inclination of the lunar orbit) distant from the pole of the ecliptic,[2] and the small circle is so exceedingly small as to produce no retrograde motion, which is also the case with the Sun. The periods of the poles of the outer planets are given by the following figures Saturn makes 57 revolutions in 59 years and $1\frac{1}{4}+\frac{1}{4}$ days, in which period the mean pole lags behind 2 revolutions $1\frac{2}{5}°+\frac{2}{5}°$. Jupiter makes 65 revolutions in 71 years, the mean pole lagging behind 6 revolutions. Mars makes 37 revolutions in 79 years and $3\frac{1}{4}+\frac{1}{15}$ days, the pole lagging behind 42 revolutions and $3\frac{1}{4}°$[3].

In other words, the motion on these small circles are completed in the synodic periods of planets. Similarly, the pole of Venus makes 5 revolutions in the 8 years less $2\frac{1}{4}d+\frac{1}{10}$ lagging $1\frac{1}{4}$ revolutions in one year ; and Mercury 145 revolutions in 46 years and $1\frac{1}{10}d$[4]. It is curious that Alpetragius alters the order of the planets, placing Venus between Mars and the Sun, because the defectus (lagging) of Venus smaller than that of the Sun.[5] He also says that nobody has given any valid reason for accepting the usually assumed order of the planets, and that Ptolemy is wrong in stating that Mercury and Venus are never exactly in a line with the Sun (a remark already made by Geber) ; and as they shine by their own light they would not appear as dark spots, if passing between us and the Sun. That they do not receive their light from the Sun is proved, he thinks, by the fact that they never appear crescent-shaped.[6]

There is no need to dwell any longer on this quaint theory

of spiral motion, as it has been rather improperly called[1] It represented a retrograde step of exceedingly great magnitude, totally unjustified as the theory could not seriously pretend to be superior to the Ptolemaic system which had only become so very simple if one was content with representing only the principal phenomena We are told by the Jewish astronomer Isaac Israeli of Toledo, that the new system made a great sensation, but that it was not sufficiently worked out to be taken seriously, and that the system of Ptolemy, founded on the most rigorous calculations, could not be superseded by it[2] Another Jewish author, Levi ben Gerson, in a work written in 1328, entered into a lengthy refutation of the hypotheses of Al Betrugi[3] But the latter certainly represented a general desire on the part of the Spanish Aristoteleans to overcome the physical difficulties in accepting the Ptolemaic system, thus Averroes says that the astronomy of Ptolemy is merely a convenient means of computing, and that he himself in his youth had hoped to prepare a work on the subject

Nasir ed din Al Tusi

While ineffectual attempts were being made in the far west to devise a new astronomical theory, the astronomers of the east did not remain blind to the desirability of finding a system, in which the planets were not supposed to move unsupported in space in such a wonderfully complicated manner; and in the thirteenth century we find one of the greatest astronomers, Nasir ed-din Al Tûsi advocating a system of spheres which he supposed to be more acceptable than excentrics and epicycles[4] In addition to a review or digest of the *Syntaxis* of Ptolemy he wrote a shorter work entitled *Memorial of Astronomy*, in various

1 e.g by Riccioli *Almag. Nov* T I, p 504 where Kepler's figure of the real motion of Mars in space from 1580 to 1596 (supposing the earth to be at rest) is copied as if that had anything to do with the "*Spirals*" of Alpetragius

2. He adds that he was not qualified himself to sit in judgment on the proposed system (*Liber Jesod Olam* II, 9 p XI)

3. Munk *Melanges* pp. 500 and 521

4. *Les sphères celestes selon Nasir Eddin Attûsi*. Par M Carra de Vaux" Appendix VI. to *Tannery's Recherches sur l'astr anc* pp 337–360 Includes a translation of the chapter in which the new theory is set forth.

centre of the Earth, and another sphere (4) with a diameter twice as great Finally (4) is placed in the interior of a carrying sphere (5) concentric with the world and occupying the concavity of the

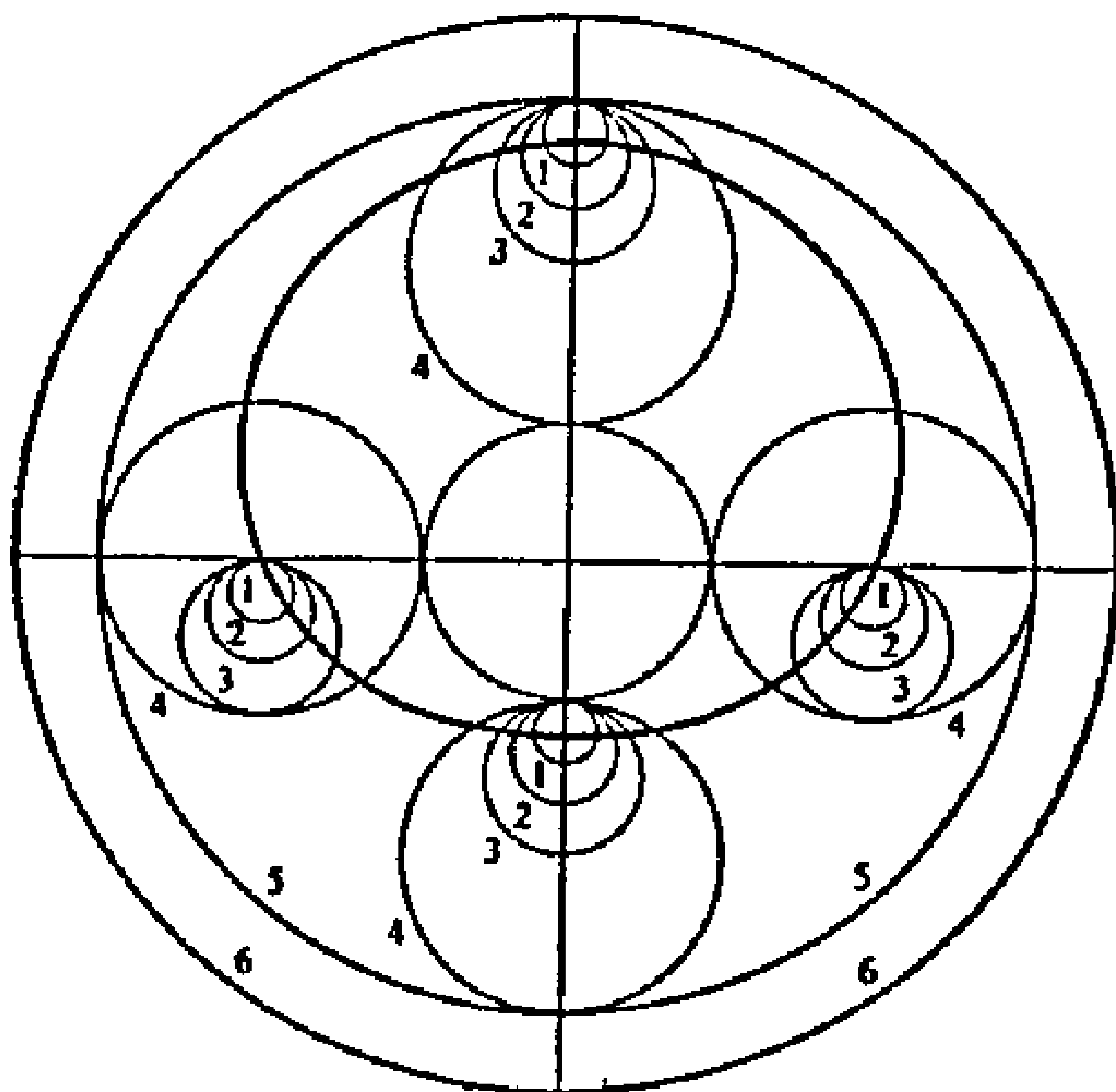


Fig 3—Movements of deferents and epicycles of planets.
The thickline is not a circle. All others are circles.

sphere (6) the equator of which is in the plane of the lunar orbit. (2) and (4) and (5) revolve in the same period that in which the centre of the epicycle performs a revolution. (3) revolves in half that time, while (6) revolves in the opposite direction with the same speed as the apogee of the excentric. The figure now shows how the epicycle moves to and fro along the diameter of (4) and during the revolution of the circle (5) describes a closed curve about which Nasir ed-din justly says that it is somewhat like a circle but is not really one, for which reason it is not a perfect substitute for the eccentric circle of Ptolemy He estimates the greatest difference between the lunar places given by the two theories as one-sixth of a degree half-way between syzygy and quadrature. Except for the action of the guiding sphere (2) it would not be the centre of the epicycle but the point of contact of circles (3) and (4), which describes the curve resembling a circle The same

method may be adopted for Venus and the three outer planets, and Nasir ed-din promises to explain the new theory of Mercury in an appendix but this appears to have been lost.

Nasir ed-din also endeavours to improve on the machinery proposed by Ptolemy to illustrate the manner in which the epicycle remains parallel to the plane of the ecliptic. He mentions that the celebrated Ibn al Haitham (afterwards known in the west as Alhazen author of a well known book on optics) had written a chapter on this subject adding to each epicycle two spheres to account for the inclination of the diameter perigee-apogee and two additional ones for the inferior planets for the diameter at right angles thereto[1] Nasir ed-din makes use of the same principle which guided him in his demonstration about the motion in longitude and he shows how in this way we may by means of two spheres make the extremities of the diameter of the epicycle move backwards and forwards along an arc of a sphere[2] He claims that this arrangement is superior to that of Ptolemy by not introducing any error in longitude[3] but he acknowledges that he has *not been able to get rid of the strong objection to Ptolemy s* auxiliary circle viz that the irregular motion in longitude with regard to the centre of the deferent necessitates the introduction *of a corresponding irregularity in the motion on the auxiliary* circle by letting the motion be uniform with regard to an equant It baffled Nasir ed-din s ingenuity to find an arrangement of spheres which could obviate the necessity of having recourse to this expedient.

All the attempts at rebellion against the Ptolemaic system *had thus turned out failures And they deserved nothing else,* since it was impossible to find anything better than what Ptolemy had produced until it was perceived that where Ptolemy was wrong was not in his mathematical methods which were perfect, but in the fundamantal idea of the Earth being at rest The time

1 Ibn al Haitham sa d that by using d scs instead of spheres one might complete the demonstration but Nas r ed-din objects to the arrangement (about which he g ves no details) that a non phetical system is not in *accordance with the principles of astronomy*

2. It is not qu te clear whether this plan is his own or is the same as Ibn al Haitham s

3 Due to disturbance of the pos tion of the diam ter from perigee to apogee from whi h the anomaly is counted

after all not be very surprising if some learned Jews had been influenced by the opinion of Herakleides, since it is an established fact that the doctrines of the Kabbalists were intimately connected with the later Greek philosophy. But any how nothing came of this isolated case, and the daily rotation of the heavens continued to be universally accepted as a self-evident fact.

Arabian astronomers and Ptolemaic system—

Arabian astronomers who really wished to follow in detail the celestial motion were therefore obliged to adopt the Ptolemaic system altogether. New planetary tables had long been found to be a necessity, and this important work was at last undertaken by King Alfonso X. of Castille and several Jewish and Christian astronomers working under him at Toledo, who prepared the *celebrated Alfonsine Tables, Apparently the King must have had his doubts about the physical truth of the system,* judging from his well known saying that if God had consulted him when creating the world, he would have given Him good advice. The tables were prepared under the direction of the Jew Ishak ben Said, called Hasan, and a physician, Jehuda ben Mose Cohen, and were finished in 1252, the year in which Alfonso ascended the throne of Castille. They continued in great repute for three hundred years as the best planetary tables; they were first printed in 1483, but had been spread all over Europe long before that time in numerous MS copies, many of which are still in existence. Twenty-six codices are counted up in the *Libros del Saber de Astronomia del Rey D. Alfonso X.de Castella,* Madrid, 1863-67 (5 vols. fol) This compilation, a series of chapters on spherical and theoretical astronomy followed by tables, must have been made up from several codices, as there are numerous repetitions even of very elementary matters. In the third volume the theories of the planets are dealt with, but one looks in vain for any improvement on Ptolemy, on the contrary, the low state of astronomy in the Middle Ages is nowhere better illustrated. In general the elements of the orbits are these of Ptolemy, though sometimes only approximations are given, while different values are given in different chapters, though Ptolemy places the centre of the deferent midway between the centre of the equant and the Earth, the *Libros del Saber* places the centre of the equant (*cerco del alaux.*[1]) midway between the Earth and the centre of the deferent

1. *al is the Arabic article, aux (apside) is a corruption of the Arabic Oudj* (Abu 'l Faraj II. p. 25). The equant is called the *cerco del Yguador.*

Though the somewhat confused collection of essays entitled the *Libros del Saber* would not, if published in the thirteenth century, have advanced astronomical science, it cannot be denied that the Alfonsine Tables were very useful in their day. The actual elements are not given, nor is any thing said about any observations by which somewhat more correct values of the mean motions must have been found.[1]

Arabs on motions of fixed stars.

Thus we finish our review of the planetary theories of the Arabs. Now we shall say a few words about their ideas as to the nature and motion of the fixed stars. The exaggerated notion which prevailed before the invention of the telescope with regard to the apparent angular diameters of the stars naturally, led to erroneous estimates of their actual size, founded on the assumption that the sphere of the fixed stars (the eighth sphere) was immediately outside that of Saturn.[2] The stars of the first magnitude were supposed to have an apparent diameter equal to $\frac{1}{20}$ of that of the Sun, from which it followed that their actual diameters were about $4\frac{3}{4}$ times that of the Earth, or about

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propter motus supradictos non (ut in aliis planetis fit) circumferentiam deferentis circularem, sed potius figuræ, habentis similitudinem cum plana ovali peripheriam describere." Next by Albert of Brudzew in 1482 in his *Commentariolum super theoricas novas*, printed at Milan in 1495 (ed. Cracow, 1900, p. 124), where it is remarked that the centre of the lunar epicycle describes a similar figure. This is also stated by E. Reinhold in his commentary to *Purbach*, 1542, fol. p. 7 verso (ed. of Paris, 1558, fol. 78) by Vurstisius in his *Questiones novae in theoricas*, & c., Basle, 1573, p. 233, and in Riccioli's *Almagestum novum* T. I, p. 564. The last three writers (who give a figure) also take the equable angular motion round the centre of the equant into account, which centre lies on the point of the *circumference of the small circle nearest the Earth. The curve described* by the centre of the epicycle thus becomes *egg-shaped*, and not like an *ellipse*.

1. The tables in vol. v of the *Libros del Saber* are quite different from the Alfonsine Tables, and are apparently only intended for astrological purposes.
2. Al Battani (cap. 50) gives the greatest distance of Saturn = 18,094 and the distance of the fixed stars = 19,000 semidiameters of the Earth. Al Fargani (p. 82) puts them exactly equal. Al Kusgi gives the diameters in parasangs, of the concavity of the stellar sphere = 33,509,180 of the ninth sphere 33,524,309, of its convexity "no one but God knows" (Shah Cholgi, p. 97).

He imagines a fixed ecliptic (in the ninth sphere) which intersects the equator in two points (the mean equinoxes) under an angle of 23° 33′ 30″, *and a movable ecliptic (in the* [illegible] *sphere)*, attached at two diametrically opposite points to two small circles, the centres of which are in the mean equinoxes and the radii of which are=4° 18′ 43″. The movable tropical points of Cancer and Capricorn *never leave the fixed ecliptic*, but *move to and fro to* the extent of 8° 37′ 26″, while two points on the movable ecliptic 90° from the tropical points move on the circumferences of the small circles, so that the movable ecliptic rises and falls on the fixed one, *while the points of intersection of the equator and the* movable ecliptic advance and recede to the extent of 10° 45′ either way. This is a motoin of the eighth sphere, common to all stars, and the Sun will, therefore, sometimes reach its greatest declination in Cancer, sometimes in Gemini. Tabit does not say that the obliquity of the ecliptic is variable, and perhaps it did not occur to him that this would be a necessary consequence of his theory, he only notices the change in direction and amount of the motion of the equinoxes, which, he says, has increased since the days of Ptolemy, when it was only 1° in 100 years, while later observers have found 1° in 66 years. The erroneous value given by Ptolemy was, therefore, mainly responsible for the continuance of the imaginary theory. It is to be observed that Tabit expresses himself with a certain reservation, and seems to think that further observations are necessary to decide if the theory is true or not. His younger and greater contemporary Al Battani was even more cautious, for though he repeats the account of the trepidation given by Theon (which he says that Ptolemy *manifeste in suo libro declarat*[1]) he does not make use of it, but simply adopts 1° in 66 years (or 54″.5 a year), which he finds by a comparison between his own observations and some made by Menelaus. *In rejecting the erroneous value of* Ptolemy, which Al Fargani alone had accepted.[2] Al Battani was followed by Ibn Jûnis, who came still nearer to the truth by adopting 1° in 70 years or 51″.2 a year, and who does not allude to trepidation. It is greatly to the credit of several other Arabian writers that they were not led astray by this imaginary phe-

1. Cap. 52 (205) Plato's translation gives the period as 84 years, but Nallino's ed. has 80 (p. 127)

2. c. 13, p. 49

3. Schjellerup, *Descr. des etoiles fixes*, p. 43.

nomenon, among them are Al Sufi the author of the only uranometry of the Middle Ages[3], who followed Al Battani, also Abu 'l Faraj and Jagmini,[1] while Nasir ed-din mentions it but seems to doubt its reality [2] By others it was willingly accepted for instance by Al Zarkali who made the period of oscillation of 10° either way equal to 2000 Muhammedan years (or 1940 Gregorian years i e 1° in 97 years or 37″ a year) The motion is in a circle of 10° radius at the Hijra the movable equinox was it 40′ in increasing precession and in A D 1080 at 7° 25 [3] The diminution of the inclination of the ecliptic, which the astronomers of Al Mamun had found=23° 33′, no doubt lent countenance to the idea of trepidation, and the next step in the development of this curious theory was the combination of progressive and oscillatory motion Al Betrugi, who gives a sort of history of the theory, beginning with a mythical H mes makes out that Theon (or Taun Alexandrinus as he calls him) combined the motion of 1° in 100 years with the oscillation[3] A century later this was actually done, and the theory received its last development by King Alfonso or his astronomers, who[4] perceived that the equinoxes had receded much further than Tabit's theory allowed The equioxes were now supposed to pass right round the heavens in 49 000 years (annual motion=26″ 45), while the period of the inequality of trepidation was 7000 years, so that in a sort of Great Jubilee year everything was again as it had been in the beginning [5] The progressive motion belongs to

1 Abu l Faraj p 12 simply says that the motion is 1° in 100 years according to Ptolemy or 1° in 66 years according to others But on p 18 he says that if the ancient Chaldeans gave the tropical points a motion backwards and and forwards, and if ancient astrologers adopted this then the motion of the fixed stars must have been unknown to them Jagmini (p 229) says that most people adopt 1° in 66 solar years

2. *Spheres celestes* p 347

3 Sedillot, *Memoire sur les instr astr des Arabes* pp 31 32. Abraham ben Chija (p 196 of Munster s *Sphaera mundi*, Basle 1546) gives the period as 1600 years without quoting any authority He adds that the ancient Ind ans Egyptians Chaldeans Greeks and Latins first proposed the theory—Ptolemy neither approved nor disapproved of it, but Al Battani confuted it.

4, Alpetragius f. 12a He says that Al Zarkali did the same

5 A later writer Augustinus Ricius *De motu octave sphaere* Paris 1521 who traces the theory back to Hermes 1985 years before Ptolemy (!) cred ts

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the ninth sphere; the annual precession varies between 26".45± 28".96, or from + 55".41 to −2".51.[1] It was now necessary to assume the existence of a tenth sphere, which as primum mobile communicated the daily rotation to all the others, while the ninth produced the progressive and the eighth periodical motion on the small circles, which are situated "in the concavity of the ninth sphere." This was a nice and comfortable theory on account of the long periods involved and the slow changes it produced in the amount of annual precession; and *oblivious of the fact that the theory had no foundation except* the circumstance that the obiquity of the ecliptic was now about 20′ less than it had been stated to be by Ptolemy, and that he had given the amount of precession as 36" a year instead of about 50", and *often shutting their eyes to several of the necessary* consequences of it, such as the changes in the latitudes of stars which it ought to produce[2], astronomers continued to accept the theory until at last a real observer of the stars arose and wiped it out by showing that the obliquity of the ecliptic had steadily diminished, and that the amount of annual precession had never varied. We have in this place only alluded to it because it in*volved some rearrangement of the spheres and because it is* eminently characteristic of the period during which no persistent observations were taken, and hardly an attempt was made to improve the theories of Ptolemy. The theory of *trepidatio* or *titubatio*, as it was sometimes called, was one attempt and it would have been better left alone. But it forms a not uninteresting chapter in the history of astronomy.

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this development to a Jew of Toledo, Isaac Hassan (see above, p 38), *adding that Alfonso four years after the completion of the tables became* convinced of the futility of the theory by reading the book on the fixed stars by Al Sufi. Riccioli, *Almag. novum*, I, p. 166

1. *In the Alfonsine* Tables *the maximum took place at the birth of Christ.* In Easler's *Speculum astrologicum*, p. 224 (appended to Purbach's *Theoricae novae*, Basle, 1573) the epoch is A.D 15, diebus 137 complens Reinhold in his commentary to Purbach (Paris, 1558, f. 163b) explains that 26".45 is the space passed over by the Sun in 10 mins. 44 secs., by which amount *the Alfonsine Tables made the tropical year smaller than 365¼ days.*

2. Abraham ben Chija (p 100, *Schrackenfuchs*) says that trepidation does not change the latitudes. *Perhaps he refers to the earliest form of the* notion, that described by Theon of Alexandria.

Here we finish our review of ancient astronomy We have omitted as not coming within our province several valuable contributions to science which did not deal with cosmology or planetary theory But even with this limitation enough has been said to show that when Europeans again began to occupy themselves with science they found astronomy practically in the same state in which Ptolemy had left it in the second century But the Arabs had put a powerful tool into their hands by altering the calculus of chords of Ptolemy into the calculus of sines or trigonometry, and hereby they influenced the advancement of astronomy in a most important manner

References

1 Peter Doig *A Concise History of Astronomy* London 1950

1 J L E Dreyer *A History of Astronomy from Thales to Kepler Dover publications* 1953 (chapter XI reproduced)

3 Satya Prakash *Founders of Sciences in Ancient India*, Delhi 1965

CHAPTER II

Personal References of Brahmagupta

Sudhākara Dvivedi in his *Ganaka Taraṅgiṇī* a small book on biographical skethes of astronomers and astrologers of this country gives a brief account of Brahmagupta thus

Brahmagupta was born in 520 Śaka (655 Vikramī or 589 A D) in the reign of King Vyāghramukha belonging to the *Cāpa* family his father was Jisnugupta and at the age of 30 he wrote in 550 Śaka (628 A D) his well known treatise on Astronomy known as the *Brāhmasphuṭasiddhānta* which is corroborated by the statement in the *Viṣṇudharmottara Purāna* (Chapter on the *Brahma siddhānta*) His other treatise entitled the *Khaṇḍa-Khādyaka* which is a *karana* book was completed in 587 Śaka (665 A D) According to some authorities Brahmagupta was the grandson of Visnugupta and the family suffix (*Gupta*) indicated that he belonged to the Vaiśya family, and he was in the service of the King of Rewah, known as Vyāghrabhaṭa

Brahmagupta was a great critic he did not spare any of his predecessors like Āryabhata Varāhamihira Śrīsena Viṣṇucandra and others Later on his influence on the writing of the succeeding generations has been immense Bhāskarācārya II in his *Bijagaṇita* has acknowledged him as a great authority on algebra and has given him as the first place amongst the galaxy consisting of Brahmagupta Śrīdhara Padmanābha etc The Eighteenth Chapter of the *Brahmasphuṭasiddhānta* known as Kuṭṭaka Chapter (on Pulveriser) has been translated by H T Colebrooke in English in 1817 The English translation of the Twelfth Chapter on Ganita or Calculations from the *Brāhmasphuṭasiddhānta* is also available in English (See Colebrook s *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāskara London 1817*)

The *Vasanā Commentary* on the Brāhmasphuṭasiddhānta by Pṛthūdakasvāmī (860 A D) is also available though with diffi culty (as indicated by Sudhākara Dvivedi), its incorrect manus-

cript is available in the Library of the King of Banaras (Kāśirāja) which has the colophony at the end as :

श्री चापवंशतिलके श्री व्याघ्रमुखे नृपे शकनृपाणाद् ,
पञ्चाशत्संयुक्तै वर्षशतैः पञ्चभिरतीतैः ।
ब्राह्मस्फुटसिद्धान्तः सज्जनगणितज्ञगोलवित्प्रीत्यै,
त्रिंशद्वर्षेण कृतो जिष्णुसुतब्रह्मगुप्तेन ॥

Bhāskara II has written the famous treatise *Siddhāntaśiromaṇi* (1150), which is almost based on the *Brāhmasphuṭasiddhānta*. It has been edited by the author's own gloss (*Vāsanābhāṣya*) by Bāpu Deva Śāstrī (Vārānasī); by Murlidhar Jha with the commentaries, *Vāsanāvārttika* of Nṛsiṁha (1621) and Marīci of Muniśvara (1635), vol. I (containing chapter I of the *Gaṇitādhyāya*) (Vārāṇasī, 1917); by Girija Prasad Dvivedī with original commentaries in Sanskrit and Hindi, vols. I and II (Lucknow, 1911, 1926); English translation of the text only by Bāpu Deva Śāstrī and Wilkinson (Calcutta, 1861).

In the very first Chapter (verse 2), Brahmagupta writes: The old calculations dealing with planets (i.e. the old astronomy), based on the system of Brahmā have become erroneous in course of past ages and therefore, I, the son of Jisnugupta would like to clarify them.

Brahmagupta was not a mere theorist, he based his calculations on direct observations with the help of instruments or devices (*nalikādi yantra*); he was in favour of making corrections on the basis of these observations. He was himself an expert *observer*. In his *Khaṇḍakhādyaka* also he has emphasised the need of direct observation.

At many places, Brahmagupta has severely criticised the Romaka and Pauliśa systems of astronomy which were introduced in this country by Lāṭadeva and Śrīṣena. There are many passages where this criticism would be available with vehemance.

Brhamagupta was opposed to the system of Āryabhaṭa I. He never spares the school of Āryabhaṭa which was regarded as the most authoritative then Sudhākara Dvivedi says that as Brahmagupta was opposed to the system of Āryabhaṭa, so the Vaṭeśvara Siddhānta was opposed to that of Brahmagupta. The Institute has already published the *Vaṭeśvara Siddhānta* and now it has the privilege of publishing the *Brāhmasphuṭasiddhānta*.

A Note on Bhillamāla

It is said that Brahmagupta completed his *Brāhmasphuṭa-siddhānta* in Śaka 550, and he has come to be known as Bhilla-mālakācarya or a teacher residing in "Bhillamālaka." In this connection, therefore, it would be interesting to reproduce a note on *Bhillamāla* from G. Bhūler's article on *Gurjara Inscriptions*, No III, published in the *Indian Antiquary*, July 1888, vol. 17, p 192 :

With a single exception all the complete inscriptions call the princes enumerated above, scions of the Gurjara race; and *Khe* I. and II. highly extol the greatness and wide extent of this family. *Na.* alone names the Mahārāja Karna as their ancestor. With respect to this personage it is for the present impossible to say whether the famous hero of the *Mahābhārata* may be meant, or some real historical king. But the name *Gurjara* makes it evident that this dynasty belonged to the great tribe which is still found in Northern and Western India and after which two provinces, one in the Bombay Presidency and one in the Pañjāba, have been named The Gurjaras or *Gūjars* are at present pretty numerous in the western Himālaya, in the Pañjāba and in Eastern Rajpūtānā. In Kachh and Gujarāt their number is much smaller. It would, therefore, seem that they came into Western India from the north. Their immigration must have taken place in early times, about the beginning of our era or shortly afterwards. In Western India they founded, besides the kingdom of Broach, another larger state which lay some hundred miles further north. Hiuen Tsiang mentions in his travels[1] the kingdom of *Kiu-che-lo* and its capital *Pi-lo-mi-lo*. It has been long known that the former word corresponds to Gurjara

But the name of the town has been incorrectly connected by the French scholars with Bālmer in the Jesalmir territory, and this identification has been accepted in Mr Beal's new translation of *Siyuki*. As I have stated already formerly[2] following Colonel J. Watson, *Pilomilo* corresponds exactly to Bhillamāla

1. Beal *Siyuki*, Vol. II, p 269f. Hiuen Tsiang assigns to the northern Gurjara State an extent about double of that given for the kingdom of Broach

9, *Ante*, Vol VI P. 63

one of the old names of the modern Bhinmāl or Śrīmāl[1] in southern Mārvāḍ close to the northern frontier of Gujarāt Another work which was composed a few years before Hiuen Tsiang's visit to Gujarāt contains likewise a notice of this northern kingdom of the Gurjaras The astronomer, Brahmagupta who completed his *Siddhānta* in Śaka Saṃvat 550 or 628 A D calls himself Bhillamālakakācārya[2], the teacher residing in Bhillamālaka and is called so by his commentator Pṛthudakasvāmin He further states that he wrote under king Vyāghramukha who was an ornament of the Cāpa race' This family, whose name recurs in the Haḍḍāla grant of Dharanīvarāha[3] prince of Vadhvān thus seems to have been the reigning house of Bhillamāla It is most probably identical with the Cāuḍās Cāvoṭakas[4] or Chāpótkaṭas who from 756 to 941 A D held Aṇhilvāḍ and still possess various small districts in northern Gujarāt The Gurjara kingdom of Broach was without a doubt an offshoot of the larger State in the north and it may be that its rulers too belonged to the Cāpa family

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1 Bhillamāla means etymologically *the field of the Bhil* and Śrīmāla the field of Srī The latter name must also be ancient as the Śrīmāli Brāhman as are called after it The Jainas narrate various of course *incredible legends which explain how Śrīmāla came to be called Bhillamāla* Merutunga says that king Bhoja invented the latter name because the people of Śrīmāla let the poet Māgha die of starvation According to *another authority the town had a different name in each Yuga It is in* India very common for ancient towns to have two or even more names Thus Kanauj was called Kānyakubja Gādhipura and Mahodaya

2 See Professor A Weber *Die Sanskrit und Prakrit Handschriften der Berliner Bibliothek* Vol. II pp 297 298 In the first passage the MSS offers incorrectly Bhillamācārya in the second which occurs in the commentary *on the Khaṇḍakhādyaka we have Bhillamālavakācārya a slightly corrupt* reading This latter varia lectio occurs also in other MSS see Weber *Indische Streifen* Vol III p 90 and has given rise to erroneous *suppositions regarding Brahmagupta's home The Gujarāti Joshis still pre*serve the tradition that Brahmagupta was a native of Bhinmāla

3 Ante Vol. XII p 190ff The remark which I have made there *that the Cāpas are not named elsewhere, of course requires correction*

4 The form Cāvoṭaka which occurs in Dr Bhagavanlal's grant of the Gujarāt Cālukya king Pulakeśin of Samvat 490 is the immediate predecessor of the word Cāuḍā Its Sanskrit original is certainly not cāpotkaṭa which probably has been coined in comparatively speaking modern times in order to explain the difficult Prakrit word just as the bards of Rajputana have invented Rāṣṭrauḍha as etymon for Rāṭhoḍ

Brahmagupta's own References

In the Twenty fourth Chapter (*Saṁjnādhyāya*) of the *Brāhmasphuṭasiddhānta*, Brahmagupta has made a reference to his own biography In the reign of Vyāghramukha belonging to the family of Cāpa in the year 550 Śaka the treatise *Brāhmasphuṭasiddhānta* was composed for the benefit of benevolent astronomers by Brahmagupta son of Jisnugupta at the age of 30 (*BrSpSi* XXIV 7 8)

Then again he says The *Brāhmasphuṭasiddhānta* has been written by Brahmagupta son of Jiṣnu in 1008 verses of Āryāchanda (ibid 10)

In the beginning of this Sanjnādhyāya he refers to the differences in fundamental notions created by the various existing systems of astronomy as the *Sūrya siddhānta Pulisa siddhānta Romaka siddhānta Vasiṣṭha siddhānta* and other *Yavana siddhāntas* which have caused anomalies in the calculations of eclipses He also refers to the anomalies due to the calculations based on midnight day reckoning and sunrise day-reckoning

From the point of view of own references the following would be of interest

Brahmagupta son of Jiṣṇugupta (Jiṣnusuta Brahmagupta)

BrSpSi I 2 XVI 35 37 XXIV 8 10 XXV 73

It is strange that in the *Khaṇḍakhādyaka* Brahmagupta has not given his name nor his father s name anywhere At least the reading of the *Khaṇḍakhādyaka* as given by Pṛthūdakasvāmi does not contain this name In the edition of Bhaṭṭotpala there are three more chapters in the *Khaṇḍakhādyaka* (Chapters IX X and XI) In the Chapter XI (known as *Paṭādhikāra*) we have 21 verses and in the last 21st verse we find the name of Brahmagupta[1] son of Jisnu mentioned

> Those who are eager to have the knowledge of the motion of stars and planets for them and for the benefit of disciples in this field Brahmagupta son of Jisnu has composed this *Khaṇḍakhādyaka*

1 खण्डखाद्यकमिदं तृप्त्यर्थ ग्रहगतिज्ञुभार्त्तानाम् ।
शिष्याणां हितार्थं प्रोक्त जिष्णुसुतब्रह्मगुप्तेन । —*KK XI 21*

Reference to Āryabhata

We have said that it appears that Brahmagupta was a bitter opponent of Āryabhaṭa in his younger days (628 A. D.), but later on (in 665 A. D.), he climbed down to describe and teach one of the Āryabhaṭa's system of astronomy. Āryabhaṭa was universally revered. and it was difficult for Brahmagupta to have ignored him and thus he has to refer to this great authority some times to oppose some of his views and some times to expound his views. The following are the pasages in the *Brahmqsphuṭasiddhanta*, where the author has referred to the name of Āryabhaṭa. There are many other passages where the name "Āryabhaṭa" does not occur but where Brahmagupta indirectly means to quote the views of this great master.

BrSpSi: I. 9. 12. 28. 32. 60. 61; II. 33. 46. V. 21.25. VI. 13; IX.10; XI. 4. 9. 10. 12 25 29. 33. 41. 42. 43. 44, 46. 47. 49. 62; XIII, 27; XIV. 45, XVI. 37. 46; XXI. 39.

The largest references are in Chapter XI, where Brahmagupta has made an attempt to show the discrepancies of the Āryabhaṭiaya system of reckoning astronomical observations and constants.

In the *Khandakhadyaka* also there is a reference to the name of Āryabhaṭa but on very few occasions. In Chapter I, we have this reference at three places-

> Having made obeisance to God Mahādeva, who is the great cause of this world's rise (i.e. creation), existence and destruction. I shall declare the *Khaṇḍakhadyaka* (i.e. a short treatise on astronomy which is as pleasant as food prepared with sugarcandy), which will yield the same results as the great astronomical treatise of Āryabhaṭa. As in most cases calculation by the great work of Āryabhaṭa, for (the knowlege of time and longitude of planets etc. at) marriage, nativity and the like, is impracticable for common use every day, this smaller treatise is made so as to yield the same results as that [1]

1. प्रणिपत्य महादेवं जगदुत्पत्तिस्थितिप्रलयहेतुम् ।
वक्ष्यामि खण्डखाद्यकमाचार्यार्यभटतुल्यफलम् ॥
प्रायेणार्यभटेन व्यवहारः प्रतिदिनं यतोऽशक्यः ।
उद्वाहजातकादिषु तत्समफललघुतरोक्तिरतः ॥ —KK. I. 1-2

These two verses show that Brahmagupta was not hostile to Ārya bhaṭa when he wrote this *Khaṇḍakhādyaka*; he merely intends presenting the subject matter on simple lines and furnishing the results obtained by Āryabhaṭa in a simpler way

In the following verse of the same chapter he refers to *Āryabhaṭa's midnight day-reckoning system*

> The mean Saturn diminished by 3 seconds, the *Śīghrocca* of Mercury diminished by 22 seconds the mean Mars increased by 2 seconds and the mean Jupiter increased by 4 seconds are equal to the respective mean planets of Ārya bhaṭa's midnight system [1]

In the Appendix of the *Khaṇḍakhādyaka* known as *Khaṇḍakhādyakottara*, we again find a few verses where the name of Āryabhaṭa occurs Three of these *verses have been reproduced* from the *Brāhmasphuṭasiddhānta (BrSpSi I 62, 63 II 47)*

> Āryabhaṭa made the apogee of the Moon as moving more quickly and the node as moving more slowly than their actual motions if his constants give correct results in relation to the end of *tithis* [i e conjunction etc] or eclipses they must be considered as *accidental as are the letters cut into wood* by weevils (*KK* IX 1 *BrSpSi* I 62)
>
> On seeing me who possess the most accurate knowledge of mean motions men who have learnt from the works of Śrīṣeṇa Viṣṇucandra and Āryabhaṭa cannot face me in any meeting just like deer on seeing a lion (*KK* IX 2 *BrSpSi* I 63)
>
> As the apparent planets beginning with Mars as derived from the works of Śrīṣeṇa Āryabhaṭa and Viṣṇucandra are far deviated from their true places the works of these authors are therefore not valued among the learned [2] (*KK* IX 3 *BrSpSi* II 47)

1 त्रिभिः शनिर्विहीनो द्वाविंशत्या बुधे ऽधिको द्वाभ्याम् ।
चतसृभिरधिको जीवोऽर्द्धरात्रिकार्यभटमध्यसमः ॥ —*KK* I 7

2 स्फुटमार्यभटः शीघ्रगमिन्दूच्चं पातमल्पगं स्वगतेः ।
तिथ्यन्तग्रहणानां घुणाक्षरं तस्य संवादः ॥१॥
मध्यगतिं वीक्ष्य श्रीषेणार्यभटविष्णुचन्द्राणां ।
सदसि न भवन्त्यभिमुखाः सिंहं दृष्ट्वा यथा हरिणाः ॥२॥
दूरभ्रष्टाः स्पष्टाः श्रीषेणार्यभटविष्णुचन्द्रेषु ।
यस्मात् कुजादयस्ते विदुषां नैवादरस्तस्मात् ॥३॥ —*KK* IX 1 3

In the Appendix of the *Khaṇḍakhādyaka*, there is another verse which also speaks in the same strain against Āryabhaṭa (this verse does not occur in the *Brāhmasphuṭasiddhānta*) ;

> As the process of finding the apparent places of planets as given by Āryabhaṭa does not make them agree with observation, I shall, therefore, speak of this process. Of the Sun the apogee is at two signs and seventeen degrees (*KK.* IX. 4)[1]

Brahmagupta opposed to Śrīṣeṇa—Viṣṇucandra, Lāṭa, Vijayanandī and others

Brahmagupta was a great critic; he did not spare Āryabhaṭa, and along with him he was vehemently opposed to the doctrine of Śrīsena, Viṣṇucandra, Lāṭadeva, Vijayanandī and Pradyumna also. He was opposed to the *Romaka* and *Pauliśa Siddhāntas*, which were the systems of foreign astronomy, derived from Greece, Babylonia and other centres of learning. He did his best to resist the foreign influences on astronomy.

The following are the verses in the *Brāhmasphuṭasiddhānta*, where Brahmagupta expressed his note of discord against the systems or notions of Āryabhaṭa, Śrīṣeṇa and Viṣṇucandra :

BrSpSi. I.60, II 46, 47; X 13, 62; XI.31, 46, 47, 48-50, 55, XVI, 36, 46; XXI.39, XXII 2.

In two of the verses, he refers to Lāṭasiṃha :

BrSpSi. XI. 46, 48.

In the following verse, he refers to Ankaciti, Vijayanandī, Pradyumna and others: *BrSpSi*.XI 58.

In the *Khaṇḍakhādyaka* we do not find the names of these adversaries of Brahmagupta ; we, however, have a reference of Śrīṣeṇa, Āryabhaṭa and Viṣṇucandra in the verses already quoted, occurring in the Appendix of the *Khaṇḍakhādyaka* (*Khaṇḍakhādyakottara*), Chapter IX, 2 and 3. As we have said before, these verses of the Appendix have been reproduced from the *Brāhmasphuṭasiddhānta*. (I.62 and II.47).

Reference to Romaka and Pauliśa systems

Brahmagupta speaks of his system as if expounded by Brahmā himself for the first time, later on deteriorated, and then revived by Brahmagupta himself. The very second verse of the *Brahma*-

1. [illegible] ।
[illegible] ॥ —*KK*. *IX*. 4.

sphuṭasiddhānta (I 2) substantiates this view

The science of astronomy (or the calculations of heavenly bodies) in course of long duration became ineffective or erroneous this was revived by Brahmagupta son of Jisnu[1] (*BrSpSi* I 2)

The system of astronomy which goes by the name Brahma (*Brahmasiddhānta*) has been handed down to us in three forms (i) one is as treated by the *Śākalya Saṃhitā* (ii) one as described in prose in the *Viṣṇudharmottara Purāṇa* and (iii) the one described by Varāhamihira in the *Pañcasiddhāntikā* which recognises the *yuga* of the duration of five years Which of these three was accepted by Brahmagupta is not clear But from the measures of the number of revolutions performed by a planet in a given period (*graha bhagaṇa*) etc it is clear that Brahmagupta acknowledges the system as propounded in the *Viṣṇudharmottara Purāṇa* In his chapter *Tantra parīkṣādhyāya* or *Dūṣaṇādhyāya* he contradicts the notions of the *Vedāṅga Jyotiṣa* which accepts the *yuga* of five years

We may further emphasize the fact that Brahmagupta has not clearly detailed out the errors to which the Brahma-siddhānta succombed in course of time and how these errors were eradicated by him During the days of Brahmagupta Romaka and Pauliśa systems were getting currency in this country Reference to these two are found in the *Brāhmasphuṭasiddhānta* at several places as follows

ROMAK I 13 XI 50 XXIV 3
PAULIŚA XIV 45 XXIV 3

In fact *BrSpSi* XXIV 3, we find the line *Sūryendu Puliśa Romaka Vasiṣṭha Yavanādyaiḥ* where we have a reference to all the then existing systems *Sūrya siddhānta Indu siddhānta. Puliśa siddhānta Vasiṣṭha siddhanta* and other *Yavana siddhāntas* Just as the Sun is one so the astronomical system is also one this is a different thing that calculations in different systems may vary according to different sunrises in different places

Brahmagupta refers at one place to Varāhamihira in *BrSpSi* XXI 39 where he has been spoken of in connection with a list of

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१ ब्रह्मोक्तं ग्रहगणितं महता कालेन यत् खिलीभूतम् ।
अभिधीयते स्फुटं तज्जिष्णुसुतब्रह्मगुप्तेन ॥ —*BrSpSi* *I* 2

anti-authoritative versions of astronomical systems :

Evam Varāhamihira-Śrīsenācāryabhaṭa-Viṣṇucandrādyaiḥ

Lokaviruddhamabhihitam *Veda-smṛtisamhitābāhyam.*

At one place, he mentions the difference between the calculations based on the system of Āryabhaṭa and the Pañca-siddhāntas (Five systems) : Pauliśa, Romaka, Vasiṣṭha, Saura, and Paitāmaha. (*BrSpSi*. XIV.46).

Brahmagupta was also familiar with the Jaina systems of astronomy; for example, at one place he uses the term "Jinoktam" (i.e., one propounded by the Jainas) : *He repudiates the concept of two Suns and two Moons "Dvāvarkaindavau"*(do canda do sujja) (*BrSpSi*. XI.3) as enunciated by the Jainas.

At several places we find a reference to the *Vaśiṣṭha-siddhānta* (*BrSpSi*. XI.49, 50; XXIV.3)

Wherever, Brahmagupta has to press for his views in preference to the views of others, he uses the words *Brāhma* or *Brahmokta* : *BrSpSi*.

Brāhma : I.32; X.62, XI, 61; XVI.37

Brahmokta : II.31, 33, X.63, 69, XV.59, XVI.38

Reference

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CHAPTER III

Manuscripts of the Brāhmasphuta Siddhānta

Sudhākara Dvivedī has given an account of some of the manuscripts of the *Brāhmasphuṭasiddhānta* in his *Bhūmikā* or Introduction appended to the edition published in the PAṆḌITA, Vol XXIV, 1902 (New Series) (i) available in the Library of the Government College, Kashi (Vārāṇasī) i. e Kāśika-Rājakiya Pāṭhālaya (ii) Dr.Thibaut's Manuscript (iii) the Manuscript in possession with Yajñadatta Sharmā the Chief Astronomer attached to the Prince of Ayodhyā It is further mentioned that Dr Thibaut's Manuscript was a copy of a Manuscript available in the Deccan College, Poona The Manuscripts (ii) and (iii) were identical The Manuscripts (iii) was very faulty and incorrect

The Manuscript from which Colebrooke translated out in English the Twelfth and the Eighteenth Chapters on *Gaṇita* or Mathematics and *Kuṭṭaka* (the Chapter on Pulveriser) respectively, appeared to be different from the three Manuscripts described above The readings differed considerably. The *Kuṭṭaka*-Chapter of that book is, writes Sudhakara Dvivedī, still available in the India Office Library (See Catalogue of the Sanskrit Manuscripts in the Library of India Office, Part V p 995)[1]

1. एतत्कृतस्यास्य सिद्धान्तग्रन्थस्यैका प्रति काशिकराजकीय पाठालयतो द्वितीया डा० थिबो साहिब महाशयतस्तृतीया चायोध्यानरेशप्रधानज्योतिर्विद्यज्ञदत्तशर्मणो मया लब्धा । डा० थिबोमहाशयस्य पुस्तकं कस्यचिद्दक्षिणदेशीय (Deccan College, Poona) डेका पुस्तकस्य प्रत्यन्तरम् । इदं पुस्तकं तथा पं० श्री यज्ञदत्त पुस्तकं चैकमातृकमेव । इदं पुस्तकत्रयमपि चाशुद्धं बहु च स्खलितं चास्ति ।

यत्पुस्तकानुसारेण व्यक्ताव्यक्तयोर्द्वादशाष्टादशाध्याययोरांग्लभाषायामनुवादः कोलब्रूकसाहिबेन कृतस्तत्पुस्तकमेतत्त्रयतो भिन्नमित्यनुमीयते विभाति पाठविभेदात् । तत्पुस्तकस्य कुट्टकाध्याय सम्प्रति इण्डिया-आफिस-पुस्तकालय भवने वर्तते (See Catalogue of the Sanskrit Manuscripts in the Library of India Office, Part V, Page 995)

पृथूदककृतास्य सिद्धान्तस्य टीका या कोलब्रूकसाहिबेन भारतवर्षे समुपलब्धा सा च संप्रति लण्डननगरे इण्डिया-आफिस-पुस्तकालय भवने वर्तते । तस्या एका प्रतिलिपिर्द्वारेण श्री डा० थिबो साहिबमहाशयेनोत्पादिता सापि सम्प्रति मन्निकटेऽस्ति । कश्चो इण्डिया-आफिस-पुस्तका

(Continued on next page)

Colebroke procured a Manuscript of the *Pṛthūdaka Svāmi's Annotation on the Brāhmasphuṭasiddhānta* from somewhere in India. This is also available in the India office Library, London. Undoubtedly it appears that this Manuscript is a copy from one written in the *Maithila* script. Dvivedi describes a few characteristics of this Manuscript. For example, on the Folio 11. 7 (७) is written instead of 9 (९) At a few other places also, the same has happened. Following a visarga, (*kva* क्व) is inscribed instead of '*ka*' (क), for example refer to Folio 12. line 6. At some places instead of *visarga* (:) we find sa (स) inscribed, for example on Folio 12, line 1. Sometimes we find a *sandhi* at the *virāma* or end of a sententence; e. g in the Golādhyāya. Folio 21. line 2. *sarvamupapannamuktamakhaṇḍena* (सर्वमुपपन्नमुक्तमखण्डेन). At some places we find *nu* (नू) written in place of *nta* (न्त). In one of the Folios, *Śri Gaṇeśāya namaḥ*. श्रीगणेशायनमः is written in Maithila script.[1] Sudhākara Dvivedī prepared a copy of this Manuscript for Dr. Thibaut, and this copy was available with Sudhākara Dvivedī when he published his commentary with Text in the *Paṇḍita*. During the course of binding, on account of carelessness, many of the Folios got misarranged, and many of them got fragmented. Sudhākara Dvivedi emphasises in his Introduction the need of careful research on the arrangement of these Folios and their readings. (See *Catalogue of the Sanskrit Manuscripts* in the Library of India Office, Part V, p. 993–995).

Sudhākara Dvivedi with considerable efforts could rearrange the Text:

Golādhyāya-Bhāṣyam 1 to 45, mutilated at the beginning
Madhyamādhikāra-Bhāṣyam 45 to 59

(Continued from previous page)

भवने अस्य पुस्तकस्य पुटकबन्धनकालेऽनवधानतया पत्राण्यमंगनानि जातानि, बहुत्र खण्डितानि च सन्ति । तानि कदाचिदनुपयुक्तपनाणा मन्ये स्युरिति तेषा सम्यगन्वेषणं समुचितम् । (See Catalogue of the Sanskrit Manuscripts in the Library of India Office Part. V. Page 993-995)

1. इद पुस्तक कस्यचिन्मैथिलाक्षरैर्लिखितस्य पुस्तकस्य प्रत्यन्तरमिति निःसंशयं प्रतिभाति । ११ पत्रे, ९ इत्यस्य स्थाने ७ इति लेखात् । एवमन्यत्रापि । विसर्गात् परतः 'क' स्थाने 'क्व' इति लेखात् । यथा १२ पत्रे, ६ पंक्तौ । तथा विसर्गस्थाने 'स' इति लेखात्, यथा १२ पत्रे, १ पंक्तौ । वाक्यविरामेऽपि सन्धिकरणात् । यथा गोलाध्यायस्य २१ पत्रे, २ पंक्तौ सर्वमुपपन्नमुक्तमखण्डेन । कुत्रचित् 'न्त' स्थाने 'नू' इति लेखात् । एकस्मिन् पत्रे मैथिलाक्षरैः श्री गणेशायनमः इति लेखाच्च ।

In this, the commentary is up to verse 31 In the *Spaṣṭādhikāra* the commentary begins from verse 29 Folio 60 At the top of the Folio we have the old numbering 1 and 115

After this, 68 folios are mis arranged and there the old numbering is marked 9

After that up to 87 folio we have in the bound volume a commentary up to the verse 6 of *Tripraśnādhikāra* Here the old numbering is marked 28 Then we have the commentary up to verse 27 of the *Tripraśnādhikāra* Here the numbering of folios is marked 1 and 159 (old numbering) and 218 new After this, then we have in the bound volume commentary up to verse 33 of the Tripraśnādhikāra On the last folio the old numbering is marked 5 and 163 the new numbering 222 After this begins the commentary on the *Candragrahaṇādhikāra* from verse 4 old numbering 1 and 297 and new numbering 257 Then follows the commentary on *Sūryagrahaṇa* up to verse 23 The last old folio numbering is 36 and 232, the new numbering 292 This is the last folio-numbering of the volume in the India Office Library After this, begins the commentary on *Grahayutyadhikāra* verse 11 old numbering 1 and 164 and the new numbering 223

Then follows the *Madhyagatyuttarādhyāya* up to verse 40 old folio numbering 119 and new numbering 178 Then we have the commentary of the *Madhyagatyuttarādhyāya* beginning from the verse 45 the old folio numbering 120 and the new numbering 179 Then we have the commentary on *Tripraśnottarādhyāya* up to verse 56 Here the last old folio number is 48 and 158 and the new numbering 217

In this way we have the commentary on the multilated *Golādhyāya* mutilated *Madhyamādhikāra* mutilated *Spaṣṭādhikāra* mutilated *Tripraśnādhikāra* mutilated *Candra grahaṇādhikāra*, mutilated *Sūrya grahaṇādhikāra* mutilated *Grahayutyadhikara* *Bhagrahayutyadhikāra* *Tantra parīkṣādhyāya* (also known as *Dūṣaṇādhikāra*) *Gaṇitādhyāya* (Arithmetic) mutilated *Madhyagatyuttarādhyāya* *Sphuṭagatyuttarādhyāya* mutilated *Tripraśnottarādhyāya*[1]

1 मया महताऽऽयासेन तमपि शकलम् एव नियोजितं ।
गोलाध्यायभाष्यम् १ ४५ भागौ खण्डितम् ।
मध्यमाधिकारभाष्यम् ४५-५६ ।

(Continued on next page)

Sudhākara Dvivedī says that nowhere in the *Brāhma-Sphutasiddhānta Madhyamādhikāra*, is found the verse

Samsādhya spaṣṭataram bijam nalikādiyantreṇa
Tat-sansṛtagrahebhyah kartavyau nirṇayādeśau

(This verse has been quoted by Dvivedi in the *Gaṇaka-Taraṅgiṇī* page 19)[1]

This verse has been quoted by Dvivedi from the Translation of *Grahalāghava* by Mallāri

Dvivedi further says that in the Manuscripts available there is mentioned a Twenty fifth Chapter under the title '*Dhyānagrahopadeśādhyāya* Dvivedi thinks that this Chapter does not constitute the *Brāhmasphuṭasiddhānta* proper, which ends in fact with twenty four chapters[2] In his commentary and Edition in the *Paṇḍita*, he has published it as a separate treatise of Brahmagupta Thus he has named his Edition as

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अत्र मध्यमाधिकारे ३१ श्लोकपर्यन्तमेव भाष्यम् ।
अतः स्पष्टाधिकारस्य २६ श्लोकतट्टीकाऽऽरब्धा ५० पत्रतोऽत्र प्राचीन संख्या पत्रोपरि १, तथा १२५ ।
अग्रेऽत्र ६८ पत्रमपगतम्, अत्र प्राचीन संख्या ६ ।

ततः ८७ पत्रपर्यन्तं संलग्नग्रन्थस्त्रिप्रश्नाधिकारस्य ६ श्लोकपर्यन्त टीका । अत्र प्राचीनपत्रसंख्या २८ । ततस्त्रिप्रश्नाधिकारस्य २७ श्लोकतट्टीका, पत्रसंख्या प्राचीना १ तथा १५१, नवीना २१८ । अग्रे संलग्न ग्रन्थस्य त्रिप्रश्नाधिकारस्य ३३ श्लोकपर्यन्त टीकान्तिमपत्रप्राचीनसंख्या ५ तथा १६६, नवीना संख्या च २२२ । ततश्चन्द्रग्रहणाधिकारस्य ४ श्लोक टीकारब्धा प्राचीनपत्रसंख्या १ तथा २६७ । नवीना संख्या च २५७ । ततः सूर्यग्रहणस्य २३ श्लोकपर्यन्त टीका । अन्तिम प्राचीन पत्र-संख्या ३६ तथा २३२, नवीना संख्या च २९२ । इण्डिया-आफिस पुस्तकपुटके चेयमन्तिमपत्रसंख्या । ततो ग्रहयुत्यधिकारस्य ११ श्लोकतट्टीकारब्धा अत्र प्राचीनपत्रसंख्या १ तथा १६४, नवीना संख्या २२३ । ततो मध्यगत्युत्तराध्यायस्य ।

४० श्लोकपर्यन्तं संलग्नग्रन्थो यत्रान्तिमप्राचीनपत्र संख्या ११६ नवीना संख्या च १७८ । ततो मध्यगत्युत्तराध्यायस्य ४५ श्लोकतट्टीकारब्धा । अत्र प्राचीनपत्रसंख्या १२० नवीनासंख्या च १७६ । ततस्त्रिप्रश्नोत्तराध्यायस्य ५६ श्लोकपर्यन्त टीका । अत्रान्तिमप्राचीनपत्रसंख्या ४६ तथा १५८, नवीना संख्या च २१७ ।

खण्डितगोलाध्यायस्य । खण्डितमध्यमाधिकारस्य । खण्डितस्पष्टाधिकारस्य । खण्डित-त्रिप्रश्नाधिकारस्य । खण्डितचन्द्रग्रहणाधिकारस्य । खण्डितसूर्यग्रहणाधिकारस्य । खण्डितग्रहयुत्यधिकारस्य । भग्रहयुत्यधिकारस्य । तन्त्रपरीक्षाध्यायस्य (दूषणाधिकारस्य) । गणिताध्यायस्य पाटीगणितस्य) । खण्डितमध्यगत्युत्तराध्यायस्य । स्फुटगत्युत्तराध्यायस्य । खण्डितत्रिप्रश्नोत्तराध्यायस्य च टीका वर्तते ।

1 सर्वेष्वपि पुस्तकेषु 'संसाध्य स्पष्टतरं बीजं नलिकादियन्त्रेण' इत्यादिश्लोको मध्यमाधिकारे नास्ति । तथा गणकतरङ्गिण्यां मल्लारिकृतग्रहलाघवटीकातो धृतोऽयं (द्रष्टव्या गणकतरङ्गिणी पृ० १९) ।

Brāhmasphuṭasiddhānto Dhyānagrahopadeśādhyāyaśca or *Brāhmaphuṭasiddhānta* and *Dhyānagrahopadeśādhyāya by* Brahmagupta (1902)[1]

The manuscript of this small treatise was also mutilated, and Dvivedi took special pains in editing it, and he revised the calculations also incorporated in this treatise

The small treatise *Dhyānagrahopadeśādhyāya* must have been composed prior to the *Brāhmasphuṭasiddhānta*, since we find a verse in the last Chapter (the 24th Chapter also known as the *Saṃjñādhyāya* of the *Brāhmasphuṭasiddhānta*) verse 9 a reference to this book

How could this result be obtained in a simple way has been shown by me in the *Dhyānagrahopadeśādhyāya* of 72 Āryā verses and therefore, it is not repeated here *BrSpSi* XXIV 9)

In the *Dhyānagrahopadeśādhyāya*, we have a verse 61, which is also found in the *Khaṇḍakhādyaka* (KK I 21)

Navatithayaḥ (159) divided by *aṣṭi* (16) pañcarasāḥ (65) divided by *vasu* (8), 10 divided by 3 each multiplied by the equinoctial shadow are the (tabular differences of) ascensional difference expressed in *vināḍis* (*KK* I 21 *DhGr* 61)

This verse then indicates that the *Dhyānagrahopadeśādhyāya* has been composed after the *Khaṇḍakhādyaka* It may also be possible that the *Brāhmasphuṭasiddhānta* and the *Dhyānagrahopadeśādhyāya* were simultaneously written and the above verse (*DhGr* 61) was repeated again in the *Khaṇḍakhādyaka*

Sudhākara Dvivedi has taken the help from the commentary of Pṛthūdaka Svāmī in the Chapters on *Pāṭīgaṇita* (Arithmetic), and has quoted the examples from this commentary At many places he has corrected the readings which were mutilated in manuscripts

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1 उपलब्धमूलपुस्तकत्रये पञ्चविंशतितमेऽध्याये वस्तुतो ब्रह्मगुप्तकृतो ध्यानग्रहोपदेशाध्यायो वर्तते। सो मयात्र पृथक्त्वेन तन्नाम्ना मुद्रितः। अत्र बहुत्र स्खलितानि पदानि तानि गणितेन संशोध्य मुद्रितान्यपि सुधीभिर्विचिन्त्यानि।

अस्य सिद्धान्तस्य चतुर्विंशतितमेऽन्तिमे संज्ञाध्याये (पृ० ४०८)
गणितेन फलं सिद्धिर्ब्राह्मे ध्यानग्रहे यतोऽध्याये।
ध्यानग्रहो द्विसप्ततिरार्याणां न लिखितोऽत्र मया॥

इति नवमश्लोकेन ध्यानग्रहोपदेशाध्यायस्य रचनेतः सिद्धान्तरचनात् पूर्वं विभाति परन्तु सपाद्यंशसाधनप्रकारेण सिद्धान्तरचनाकाल एवास्य रचना सिध्यति तथा 'नवतिथयोऽष्टिविभक्ता' इत्यादि पद्येन खण्डखाद्यरचनात् पश्चात् सिध्यति।

Reference

Sudhākara Dvivedi His *Bhūmikā* on *BrSpSi*, *Pandita* Vol XXIV, 1902 (New Series)

Chapter IV

Subject Matter Classified in the Brāhmasphuṭasiddhānta & Khaṇḍakhādyaka

It needs no emphasis that Āryabhaṭa commanded a great influence as an astronomer not only prior to Brahmagupta but during his days also and we have seen how Brahmagupta quoted this great authority in his writings, sometimes borrowing from him and sometimes contradicting him or improving upon his calculations Āryabhaṭa s great work is known as the *Aryabhaṭīyam* written in 499 A D Āryabhaṭa was born in 476 A D (Kaliyuga Saṃvat 3577) The *Aryabhaṭīyam* is also known as the *Aryasiddhānta* For details about Āryabhaṭa the reader is referred to the Chapter entitled" *Aryabhaṭa lays Foundations of Algebra (Founders of Sciences in Ancient India* 1965, Chapter X) This great work was written in Kusumapura (modern Patna Bihar)

Divisions of Aryabhaṭīyam

The *Aryabhaṭīyam* is divided into four chapters called *Pada* (i) the *Gītika Pada*, with ten verses (ii) the *Gaṇita Pada* with 33 verses (iii) the *Kalakriya Pada* with 25 verses and (vi) the *Gola Pada* with 50 verses

In the *Gaṇita Pada* a chapter on mathematics, we have such subjects squares (*varga*), cubes (*ghana*) (*verse 3*), square-root (*vargamūla*) (4) cube-root (*ghanamūla*) (5), area of a triangle volume of a prism (6) area of a circle, volume of a sphere (7), area of a quadrilateral (*viṣamacaturaśra*) (8) circumference of a circle (10) Rsine (radius x sine) (*Jīva*) (11) determination *of the Rsine of the zenith distance, the base (bahu) of* a right-angled triangle and the upright (*koṭi*) of a right-angled triangle (16), hypotenuse (*karṇa*) of a right-angled triangle and *ardhajyā* (17), Reversed-sin (*śara*) (18) areas of series figures (*śreḍhīphala*) (19), rule of three (*trairāśika*) (23) reduction of fractions (*savarṇīkaraṇa* of *bhinna*) (27) inverse-rule of three (*vyasta*) (28),

evaluation of unknown values (*mūlya pradarśana* of *avyakta mūlya* (30), and the theory of pulveriser (*Kuṭṭaka*) (32, 33) In this chapter, Āryabhaṭa gives the solutions of quadratic equations and thus he earns for himself the credit of founding the science of algebra

(iii) The *Kālakriyā Pāda* with 25 verses, which enumerates the units of time [1 year (*varṣa*)=12 months (*māsa*), 1 month= 30 days (*divasa*), 1 day =60 *nāḍis*, 1 *nāḍi*=60 *vināḍi* *vināḍi* or *vināḍikā* is the same as *vighaṭikā* equivalent to our 24 seconds, *nāḍi* or *nāḍikā* or *ghaṭi* is equal to 24 minutes][1], correlation of time division with the *kṣetra*—division or angular division[2] Twelve signs of Zodiac or *rāśis* go to constitute a *bhagaṇa*[3] solar day (*ravimāsa*) lunar day (*śaśi māsa*) additional month or intercalary month (*adhimāsa*) various kinds of years the solar year is human or *manuṣya* year, 30 human years=1 *pitṛ* year, 12 *pitṛ* years=1 *divya* year (divine year) 12,000 *divya* years constitute a *yuga* (6, 7 8), the first half of the *yuga* is *utsarpiṇi kāla* and the latter half is *avasarpiṇi kāla* and they are calculated from the apex of the Moon (*candrocca*) (this is not very clear) (9), a *yuga* is of 60 years, and such 60 *yugas* that is 3 600 years had passed away since the *kaliyuga* when the author was of 23 years of age, (10), the count of a *yuga* year, month and

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१ वर्षं द्वादशमासास्त्रिंशद्दिवसो भवेत् स मासस्तु ।
षष्टिर्नाड्यो दिवसः षष्टिस्तु विनाडिका नाडी ॥

—*Ārya III 1*

२ गुर्वक्षराणि षष्टिर्विनाडिकार्क्षी षडेव वा प्राणाः ।
एवं कालविभागः क्षेत्रविभागस्तथा भगणात् ॥

—*Ārya III 2*

यावता कालेन षष्टिगुर्वक्षराण्युच्चरति मध्यमया वृत्त्या पुरुषः ।
तावान्कालः आर्क्षी विनाडिका ।, यावता कालेन पुरुषः
षडुच्छ्वासान् करोति, तावान्कालश्चार्क्षी विनाडिका स्यात् ।

(परमादीश्वर)

Just as we have the time division similarly we have the kṣetra division or the circular angular division. A year has twelve months so do we have 12 *rāśi* in a *bhagaṇa* One thirtieth of a *rāśi* is one *bhāga* one sixtieth of a *bhāga* is one *lipta* one sixtieth of a *lipta* in one *vilipta* and one sixtieth of a *vilipta* is one *tatpara*

३ चन्द्रोऽंशैर्द्वादशभिरविक्षिप्तोऽर्कान्तरस्थितैर्दृश्य ।
नवभिर्भृगुर्भृगोस्तैर्द्व्यधिकैर्यथाश्लक्ष्णाः ॥

—*Ārya IV 4*

day should begin from the month *Caitra Śukla Pratipada* (the first day of the brighter half of the month *Caitra*) (11) *mandocca* (apex of slowest motion) and *śighrocca* (apex of fastest motion) (17—24)

(iv) The *Gola Pāda* with 50 verses In Verse 1 there is a reference to a point in the Sun s path the commencement of *meṣa* (*meṣādi*) this must have been *vasanta*-equinox The ascending nodes (*pāta*) of planets and the shadow of the Earth move on the path of the Sun (*arka apamaṇḍala*) (2 3), the angular difference in relation to the Sun at the appearance of Moon (12 degree or *aṁśa*) of the Venus (9 degrees or 9 *vināḍikā*) of the Jupiter (2 more than the Venus i e 11 *vināḍikās*) the Mercury or *Budha* (13 *vināḍikās*) of the Saturn or *Śani* (15 *vināḍikās*), and of Mars or *kuja* (17 *vināḍikās*)

The half of the Earth Moon planets and stars is dark since those parts happen to be under their own shadow the other half is bright as it faces the Sun (this is not true with respect to stars—author) (5) The Earth is surrounded with an atmosphere of air and water (6, 7) In the *Brahma Divasa* (Brahma s day) the sphere of the Earth is increased by one yojana and decreased by this amount during the Brahmā s night (8) Just as a person sitting on a moving boat sees the stationary trees etc on the bank of a river moving in the opposite direction similarly the stationary stars are seen moving from Laṅkā (or equator) moving towards the West(9) On account of *pravahavāyu* (air) the *nakṣatra* system and planets rise and set receding towards the West (10) The dimensions of *Sumeru Parvata* (North pole) is given to be one *yojana* and it shines like a jewel (11) and in the next verse is given the position of Sumeru and the *Baḍavāmukha* (South pole) (12) The four cities situated at a difference of 90' each on the equator are given (13) The distance of Ujjayini from Laṅkā (thus giving the latitude of Ujjayini) is given (14)

On account of the thickness of the Earth-sphere the *Khagola* (celestial sphere) is seen less than the hemisphere (15) The next verse describes how moving appears the Khagola on the North and South poles (16) Then is given the measure of day and night of *devas* (gods) *pitar* (fathers) *asura* (demons) and *manuja* (men) (17) Then are given a few technical definitions

of celestial mathematics (18—21) like *dṛṣṭi sthāna* (intersection of horizontal and vertical axis—*pūrvaparādiggatā rekhā* and *adha ūrdhva-diggatā rekhā*) *Dṛṅ-maṇḍala dṛgkṣepamaṇḍala* (*dṛgkṣepa* is the zenith distance of that point of a planet's orbit which is at the shortest distance from the zenith) Then the *Bhūbhagola* instrument is described (22, 23) Then follow the formulæ for calculating *lagna* (the horizon ecliptic point in the East) *kāla* etc comprising the *Tripraśnādhikāra* (24—33) In the next verses we have *lambaka* or Rsine of colatitude (34) *dṛk karma* (35) and *ayana dṛk-karma* (36) Then follow the calculations of lunar and solar eclipses (37—47) Verse 48 describes the coordinates of the Sun which are determined by the conjunction of horizon with the Sun of the Moon by the conjunction of the Sun and the Moon of planets by the conjunction of the Moon and planets or stars Verse 49 describes how this jewel treatise has been procured out of an ocean of true and untrue knowledge with the help of a boat of intellect This means that the author has taken special pains in discriminating true knowledge from falsehood with respect to the prevalent notions of astronomy In the last verse he says that he has not given anything new he has given that very knowledge which was imparted by the *Svayambhu* in the earliest times (50)

The *Pañchāṅgas* prepared according to the rules and formulæ of the *Āryabhaṭīyam* are still regarded with reverence by the *Vaiṣṇavas* in the South Brahmagupta was a great critic of Āryabhata but finally he wrote his treatise the *Khaṇḍakhādyaka* on the basis of this very *Āryabhaṭīya* (this treatise is a *karaṇagrantha* i e one containing a principal element of the Indian Calendar) The four commentaries of the *Āryabhaṭīya* available in Sanskrit are of Bhāskara I Sūryadeva Yajvā Parameśvara and Nīlakaṇṭha Two English translations by P C Sengupta (1927) and W E Clark (1930) are also available

Major Landmarks

Indian astronomy which reached its zenith in the times of Aryabhaṭa and Brahmagupta shows its evolution in the following stages

1 Rudiments of astronomy in the *Vedas* and *Brāhmaṇa* Books like the *Taittirīya Saṁhitā* and the *Śatapatha Brāhmaṇa* This period is associated with the dis

covery of the Vedic era some of the planets the twenty four *nakṣatras* cycle of seasons concept of leap year the dimensions of a yuga solar and lunar years and the like

II Astronomy of the *Vedāṅga* also known as the *Vedāṅga Jyotiṣ* Lagadha is the most prominent figure of this period (1400 B C to 850 B C), he is the first compiler of a text on astronomy In his work we for the first time in history find a reference to the *Jñeya Rāśi* (the knowable or the unknown quantity) and the *Jñāna-Rāśi* (known quantity) He lays the foundations of astronomical calculations In his treatise we find a mention of such subjects as Solstices (northern and southern journey of the Sun) increase in days and nights in the *ayanas* or solstices, solstitial *tithis* omission of *tithis* *parva rāśi* acceptable and non acceptable *parvas* addition of day acceptable *parvas* concept of *yoga* (a term applied to the joint space which would be travelled by the Sun and the Moon in a given period of time on the presumption that these two bodies have travelled in directions opposite to each other) method of finding out a *nakṣatra* on any *parva* day, distinction between *parva nakṣatra* and *tithi-nakṣatra* correlation of solar and lunar dates measure of a *nadika* (unit of time) *nakṣatra of the Sun yoga and its nakṣatra parva-bhaśeṣa* and equivalent *kalas* solar year lunar revolutions (risings of *nakṣatras*) deities of *nakṣtras* lunar and *sāvana day* differences (*ahika māsa*) divisions of a *sāvana day* and length of day in two *ayanas* This author probably belonged to Kashmir

III The period of *Siddhāntas* Varāhamihira in his well known treatise the *Pañcasiddhāntikā* refers to five *Siddhāntas* or systems of Astronomy Paitāmaha Vasiṣṭha Romaka Pauliśa and Sūrya (Saura) As regards its importance he gives the first place to the Sūryasiddhānta places next the Romaka and Paulisa and declares the remaining two to be definitely inferior to the former We do not possess the full *treatise of* these Siddhāntas except the *Sūryasiddhānta* Here too

we have difficulty The *Sūryasiddhānta*, as summarised by Varāhamihira in his *Pañcasiddhāntika* in many essential features differs from the system prescribed by the Text of the *Sūryasiddhānta* now available So we have two versions, the one with which Varāhmihira was familiar and the modern one

The present *Sūryasiddhānta* comprises of fourteen chapters called *adhyāyas* We have an authentic commetary on it by Parameśvara The first two chapters of the *Sūryasiddhānta* have no special name The classification of chapters is as follows

Modern Surya Siddhanta

Chapter	Name of the Chapter	Number of verses	Subject
I	—	69	Mean longitude of planets
II	—	68	True motion and true longitude of planet and elements of *Pañcānga*
III	Tripraśnādhyāya	50	Directions place and time
IV	Candra grahanā dhyāya	29	Lunar eclipse
V	Surya grahanādh yāya	17	Solar eclipse
VI	Chedyakādhyāya	24	Projection of eclipse on a plane surface
VII	Graha samāgama yuddhādhyāya	24	Conjunction of one planet with another
VIII	Tārāvisayodhyāya	21	Conjunction of a Planet with the junction star of a naksatra
IX	Udayāstamaya vi sayodhyāya	18	Heliacal rising and setting of planets
X	Candrāstamayādi visayāh	16	Moon rise and eleva tion of Moon horn
XI	Vyatipāta visayāh	23	Pāta (*vyatipāta*)
XII	—	87	Cosmogony and geo graphy

bhaṭa I s midnight system In this connection it must be remembered that Varāhamihira nowhere expresses his indebted ness to Āryabhaṭa I

The *Sūryasiddhānta* undoubtedly is the most popular book on Astronomy in this country It has been so for the last 1000 years as is seen from the list of commentators on its text (K S Sukla has given a list of 28 commentators including those who wrote commentaries in the South Indian languges like Telugu and Kannaḍa)

Allanarya Suri	Maheśvara
Amaredya	Mallikārjuna Sūri (1178 A D)
Bhaṭṭotpala (966 A D)	Nārāyana
Bhūdhara (1572 A D)	Nrsinha Daivajña (b 1586 A)
Bhūtavisnu	Nṛsiṃhadeva
Caṇḍeśvara (1178)	Parameśvara (1432 A D)
Cola	Rāghava Śarmā (1592 A D)
Dādābhai (1719 A D)	Ramakṛsna Ārādhya (1472 A D)
Devidāsa	Ranganātha (1603 A D)
Kamala Kara Daneśvarah (1618 A D)	Sārvabhauma
Kāmabhaṭṭa	Tamma Yajva (1599 A D)
Kṛsna Daivajña	Yallaya (1472 A D)
Madanapāla	Visvanātha (1628 A D)
Mādhavācārya	

A large number of astronomical books in this country were written on the basis of the *Sūryasiddhānta* as the *Gaṇakānanda* by Sūrya (1387 1447 A D), *Ganitādarśa* by Dharmapathin *Makaranda Sāriṇi* by Makaranda (1478 A D), *Grahacakra* by Kucanācārya (1299 A D) *Visṇukarana* by Viśnu (1556 A D) besides many others with indefinite dates such as the *Sūrya siddhānta nayana prakārah Sūryasiddhānta ganita Sūryasid dhānta samgraha* by Viśvanātha Sūri and Sūryasiddhānta Sārini by Rāmadatta Daivajna K S Sukla has given a list of this literature in his Introduction to the *Sūryasiddhānta*

(iv) *Period of Bhāskara* I The author or the compiler of the *Sūryasiddhānta* is not known nor its date of composition Bhāskara and Brahmagupta are the brilliant names of a contem porary period Bhāskara I lived in the seventh century of the Christian era and was a contemporary of Brahmagupta (628

A D.) *He wrote three works on Astronomy which were most* likely composed in the following order (i) the *Maha Bhāskarīya* (ii) a commentary on the *Āryabhaṭīya* and (iii) the *Laghu Bhāskarīya* His commentary on the *Āryabhaṭīya* was written in 629 i e only one year after the *Brāhmasphuṭasiddhānta* His commentary on the *Āryabhaṭīya* was written in 629 A D, i e one year after the completion of the *Brāhmasphuṭasiddhānta* Undoubtedly Bhāskara was the follower of Āryabhaṭa I Shukla says that his works provide us with a detailed exposition of the astronomical methods taught by Āryabhaṭa I and throw light on the development of Astronomy in India during the sixth and early seventh centuries A D which was the most brilliant period in the history of Indian Astonomy Shukla has brought a critical edition with English translation of the *Mahābhāskarīya* and the *Laghu Bhāskarīya* and he proposes to bring out a volume on the life and works of this great astronomer

Division of Mahabhaskariya

The *Mahābhāskarīya* is divided into Chapters called the *adhyāyas* The *adhyāyas* have not been named as some of the Chapters of the *Sūryasiddhānta* or the *Brahmaspuhtasiddhānta* The number of verses in the *Mahābhāskarīya* is as follows

	Subject	*Number of verses*
Adhyāya I	Mean Longitude of a Planet and Planetary Pulveriser	52
Adhyāya II	The Longitude Correction	10
Adhyāya III	Direction Place and Time Junction-stars of the Zodiacal Asterisms and conjunction of Planets with them	75
Adhyāya IV	The Longitude of a Planet	64
Adhyāya V	Eclipses	78
Adhyāya VI	Rising Setting and Conjunction of Planets	62
Adhyāya VII	*Astronomical Constants*	35
Adhyāya VIII	Examples	27

These titles to the subject matter given in the above table have been assigned by Dr Kripa Shanker Shukla in his critical

edition The total mumber of verses in this work is 403 The Suryasiddhanta has as stated above 500 verses in all

Laghu-bhaskariya

The *Laghu Bhaskariya* is also divided into *eight chapters* each chapter is known as *Adhyaya* The subject matter in the *book has* been dealt with as follows

Subject			*Number of verses*
Adhyāya	I	Mean longitudes of the Planets	37
Adhyāya	*II*	*True Longitudes of the* Planets	41
Adhyāya	III	Direction, Place and Time from Shadow	35
Adhyāya	IV	The Lunar Eclipse	32
Ādhyāya	V	The Solar Eclipse	15
Adhyāya	VI	Visibility, Phases and Rising and Setting of the Moon	25
Adhyāya	VII	Visibility and Conjunction of the Planets	10
Ādhyāya	VIII	Conjunction of a *Planet and a Star*	19

The total number of verses in this text is 214 The *Laghu Bhaskariyam* is thus an abridged edition of the *Maha Bhaskariyam* From the closing stanza of this work it is clear that the author wrote this work for the benefit of young students with immature mind by condensing and simplifying the contents of his larger work the *Maha Bhasariyam* (also known as *Karma-nibandha*) Thus we have a little of the parallelism Brahmagupta after finishing his bigger treatise the *Brahma sphutasiddhanta*, wrote a minor abridged work the *Khanda khadyaka* as a *karana grantha* This latter work of Brahmagupta however incorporates some original ideas not included in the earlier work Shukla has given an analytical table indicating the rules of the *Maha-Bhaskariyam* incorporated in the *Laghu Bhaskariyam* also in an abridged or modified form and also a list of the rules which have been omitted in the *Laghu Bhasakriya* There are a few rules in the *Laghu Bhaskariya* also which have no counterpart in the

Maha-Bhaskarıya Shukla further says that the arrangement of the contents of the *Laghu-Bhāskarıya* is more systematic and logical than that of the *Maha Bhāskarıya*, and is, at the same time, in keeping with the general practice followed by the other Hindu astronomers Numerous quotations of this work occur in the annotative works of Sūryadeva (b 1191 A D) Yallaya (1480 A D) Nilakantha (1500 A D) Raghunātha Rāja (1597 A D) Govinda Somayājī and Visnu Śarmā and in the *Prayoga-racanā* an anonymous commentary on the *Mahā-Bhāskarıyam* We find the commentaries of this abridged work in Malayalam and Tamil also All this speaks of the great popularity of this work

There are circumstantial evidences to show that Bhāskara I had associations with the countries of Aśmaka and Surāstra His commentary on the *Āryabhatıya* was probably written in the city of Valabhī in Surāstra Perhaps Bhāskara I was born and educated in Aśmaka and later on he migrated to Valabhī where he wrote his commentary on the *Āryabhatıya* or that he was a native of Valabhī and got his education in the Aśmaka country Perhaps there was a strong school of Astronomy in the Aśmaka country which was founded by the followers of Āryabhata so much so that at places Bhāskara I has also called Āryabhata as Āśmaka his *Āryabhatıyam* by the name *Aśmaka-Tantra* or the *Aśmakıya* and the followers of Āryabhata as Āśmakīyāḥ This Aśmaka country or Aśmaka Janapada is mentioned in the Buddhist literature also It was somewhere either in the north-west of India or was situated between the rivers Narmadā and Godāvarī Bhāskara I was evidently a resident of the latter Aśmaka (which was between the Narmadā and Godāvarī)

Brahamagupta and Bhāskara I were contemporaries Both of them developed their systems in the earlier part of the seventh century A D (3700 years of *Kaliyuga*) *Brahmasphutasiddhanta* was written in 628 A D and the commentary on the Ārya bhatīyam by Bhāskara I was composed in 629 A D Bhāskara closely followed Āryabhata but Brahmagupta had the guts to oppose the views expressed by this great master and he not only contradicted him at places but also propounded many new ideas methods of calculation and constants of greater accuracy

The classification of the contents of Astronomy in *adhikāras* appears to be the original concept of *Brahmagupta*, this system was to some extent adopted in the modern *Sūrya Siddhānta* in the case of a few chapters The *Vaṭeśvara-Siddhānta* by Vaṭeśvarācārya (born 802 Śaka or 880 A D in Ānandapura city, Punjab son of Mahādatta) also adopts the terminology *Madhyamādhikāra Spaṣṭādhikāra* and *Tripraśnādhikāra* We owe this type of caption-nomenclature to Brahmagupta

Contents of the Brahmasphuṭasiddhānta

Now we shall summarise the contents of the Chapters of this great treatise and also enumerate the number of verses in each chapter The author has himself given the total number of verses in the Chapter in the ending verse of each chapter Sometimes the verse specifying this number itself is not taken into account while giving the total number of verses in that chapter, and therefore, there occurs a minor discrepancy in the actual number and the number specified by the author himself The following table records both these numbers separately

Chapter	Title	Number of verses indicated by the author	Actual number of verses
	PŪRVA DAŚĀDHYĀYI (First Ten Chapters)		
I	Madhyamādhikāraḥ (Madhyamagati-radhyāyaḥ)	63	63
II	Spaṣṭādhikāraḥ (Sphuṭagatiradhyāyaḥ)	67	68
III	Tripraśnādhikāraḥ (Tripraśnādhyāyaḥ)	66	66
IV	Candragrahaṇādhikāraḥ (Candragrahaṇādhyāyaḥ)	20	20
V	Sūryagrahaṇādhikāraḥ (Arkagrahaṇam or Ravigrahaṇādhyāyaḥ)	26	27
VI	Udayāstādhikāraḥ (Udayāstamayādhyāyaḥ)	12	13

VII	Candraśṛṅgonnatyadhikāraḥ (Candraśṛṅgonnati adhyāyaḥ)		18	18
VIII	Candracchāyādhikāraḥ (Candracchāyā adhyāyaḥ)		9	9
IX	Grahayutyadhikāraḥ (Grahamelanādhyāyaḥ)		26	26
X	Bhagrahayutyadhikāraḥ (Bhagrahayutiḥ adhyāyaḥ)		70	70
	Total (or Daśādhyāyī)		377*	380*
XI	Tantra Parīkṣādhyāyaḥ (Dūṣaṇādhyāyaḥ)		63	63
XII	Gaṇitādhyāyaḥ			
	Miśraka vyavahāraḥ	1—16		
	Śreḍhī vyavahāraḥ	17—20		
	Kṣetra vyavahāraḥ	21—39		
	Vṛttakṣetra gaṇitam	40—43		
	Khāta vyavahāraḥ	44—46		
	Citi vyavahāraḥ	47—		
	Kākacika vyavahāre karaṇasūtre	48—49		
	Rāśi vyavahāraḥ	50—51		
	Chāyā vyavahāraḥ	52—66	66	66
XIII	Praśnādhyāyaḥ (Madhyagatyuttarādhyāyaḥ)		49	48
XIV	Sphuṭagatyuttarādhyāyaḥ		54	55
XV	Tripraśnottarādhyāyaḥ		60	60
XVI	Grahaṇottarādhyāyaḥ		46	47
XVII	Śṛṅgonnatyuttarādhyāyaḥ		10	10
XVIII	Kuṭṭādhyāyaḥ (Kuṭṭakādhyāyaḥ)		103	102
	Kuṭṭākāraḥ	1—29		
	Dhanarṇa śūnyānām saṃkalanam	30—42		
	Ekavarṇa samīkaraṇa bījam	42—50		
	Anekavarṇa-samīkaraṇa			

	bijam	51—59		
	Bhāvita-bijam	60—63		
	Varga-prakrtih	64—74		
	Udāharanāni	75—102		
XIX	Śaṅkucchāyādijñanādhyāyaḥ		20	20
XX	Chandaścityuttarādhyāyaḥ		20	19
XXI	Golādhyāyaḥ		70	70
	Sāmānya-golaprakaranam	1—16		
	Jyā-prakaraṇam	17—23		
	Sphuṭagati-vāsanā	24—30		
	Bimba-sādhanam	31—35		
	Grahana-vāsanā	36—48		
	Golabandhādhikārah	49—70		
XXII	Yantrādhyāyah		53	57
XXIII	Mānādhyāyah		12	12
XXIV	Sañjnādhyāyaḥ		13	13
		Total	*1016*	*1022*

In one of the verses, Brahmagupta states that he has composed the treatise containing 1008 verses. Sudhākara Dvivedī has given the total as 1021, whereas he says, this number according to Brahmagupta's own statement should be 1020. If one deducts the concluding 12 verses of the *Sañjñanādhyāya*, the number should be 1008.

Sudhākara Dvivedī, in his addition of the *Brahma-sphuṭa-siddhānta* (published in the *Paṇḍita*, *1901 and 1902) gives as a supplement a small treatise of Brahmagupta known as Dhyāna-grahanopadeśādhyāya or Dhyānagrahaṇādhikāraḥ* which has 72 verses.

It would be worthwhile to give here the details of the *Khaṇḍakhādyaka* also, a book of Brahmagupta about which we have spoken so much. The titles to the chapters have not been indicated in the Text ; most likely they have been assigned by the commentator. Pṛthūdaka Svāmī known as the *Khaṇḍakhādyakavivaraṇam*.

Chapter	Title	Number of verses
I	Tithi nakṣatrādhikārādhyāyah (On tithis nakṣatras etc)	32
II	Grahagatyadhyāyah (On the mean and true places of star planets)	19
III	Tripraśnādhyāyah (On the three problems relating to diurnal motion)	16
IV	Candragrahanādhyāyah (On lunar eclipses)	7
V	Sūryagrahanādhyāyah (On solar eclipses)	6
VI	Udayāstādhikārah (On the rising and setting of planets)	7
VII	Candraśṛṅgonnatyadhāyah (On the position of the Moon s cusps)	4
VIII	Samāgamādhyāyah (On conjunction of planets)	6
	UTTARA KHANDAKHĀDYAKA—APPENDIX	
IX	Corrections and new methods	14
X	On conjunction of stars and planets	16
	Total	127

Bhaṭṭotpala in his commentary on the *Khaṇḍakhādyaka* has given several additional verses in the main or proper treatise and also in its Uttara portion or the Appendix P C Sengupta s edition (Sanskrit Text 1941) has given at the end of this publication the account of these additional verses The English

edition (1934) classifies the *Uttara Khaṇḍakhādyaka* into two chapters (which the author calls as Chapters IX and X) the Sanskrit edition gives 3 verses in Chapter IX 21 verses in Chapter X and 24½ verses in Chapter XI Of these three chapters the Chapter X has been given the title *Paṭādhikāra* and Chapter XI the title *Parilekhādhyāya* by Bhaṭṭotpala

TABLE

Arrangement of contents in different treatises

Topic	*BrSpSi*	*SūSi*	*MBh*	*MSi*	*SiŚe*	*SiŚi*
Mean longitudes of the planets	I XIII	I	I	I II	I,II	I
True longitudes of the planets	II XIV	II	IV	III	III	II
Direction, place and time	III,XV	III	III	IV	IV	III
Computation of a lunar eclipse	IV,XVI	IV	V	V	V,VII	IV,V
Computation of a solar eclipse	V XVI	V	V	VI	VI	VI
Projection of an eclipse	XVI	VI	V	VIII		V
Conjunction of a planet with another planet	IX	VII	VI	XI	XI	X
Conjunction of a planet with a star	X	VIII		XII	XII	XI
Heliacal rising of planets	VI	IX	VI	IX X	IX	VII VIII

Topic	*BrŚpSi*	*SūSi*	*MBh*	*MSi*	*SiŚe*	*SiŚi*
Moonrise and elevation of lunar horns	VII,XVII	X	VI	VII VIII	X	IX
Pāta	XIV	XI	VII	XIII	VIII	XII
Cosmogony and geography	XXII	XII		XVI		II iii
Astronomical instruments	XXIII	XIII	III		XIX	II xi
Time reckoning	XXIV	XIV				I i

Āryabhaṭa and Brahmagupta Controversy

The scientific Indian astronomy was more or less created by Āryabhaṭa I (476 A D) It is said that he was the teacher of two distinct systems of astronomy one of which is called the *audayika* system and the other the *ardharātrika* system In the first the astronomical day is taken to begin at *sunrise* at Lankā and in the other the same begins at the *midnight* of the same place In the *Khaṇḍakhādyaka* Brahmagupta gives compendious rules for the calculation of longitudes etc of planets according to the *ardharātrika* system of Āryabhaṭa I In this connection he refers to Āryabhaṭa in the following words in his *Khaṇḍakhādyaka*

> Having made obeisance to God Mahādeva who is the great cause of this world s rise (i e creation) existence and destruction I shall declare the *Khaṇḍakhādyaka* which will yield the same results as the great astronomical treatise of Āryabhaṭa [1]

> As in most cases calculation by the great work of Āryabhaṭa for (the knowledge of time and longitude of planets etc at) marriage nativity and the like is impracticable for common use every day this smaller

1 प्रणिपत्य महादेवं जगदुत्पत्तिस्थितिप्रलयहेतुम् ।
वक्ष्यामि खण्डखाद्यकमाचार्यार्यभटतुल्यफलम् ॥

treatise (i e. *Khaṇḍakhādyaka*, literally meaning food prepared from sugar-candy) is made so as to yield the same results as that.[1]

The mean saturn diminished by 3 seconds, the *Śighrocca* of Mercury diminished by 22 seconds, the mean Mars increased by 2 seconds and the mean Jupiter increased by 4 seconds are equal to the respective mean planets of Āryabhaṭa's *midnight system*.[2]

In the *Brāhmasphuṭasiddhānta*, Brahmagupta accepts the astronomical day to begin with the sunrise at Laṅkā, and the calculations of days, months, years, Yugas, and Kalpas all begin from Caitra Śukla Pratipadā (the first *tithi* of the month Caitra in the bright-half of the Moon) and the first day is regarded as Sunday.[3]

Varāhamihira in his epicyclic cast to the *Sūryasiddhānta* in his *Pañcasiddhāntikā* adopts the *ardharātrika* system or the *system of reckoning days from midnight*. The question why Brahmagupta who was so bitter an opponent of Āryabhaṭa I in his younger days (628 A.D.) claimbed down to describe and teach one of the systems of Āryabhaṭa's astronomy in his sixty-seventh year (665 A D.) is difficult to explain. In fact so great was Āryabhata's reputation and fame that in spite of Brahmagupta's severe ricticisms of the former in Chapter XI of the *Brāhmasphuṭasiddhānta*, it perhaps was undiminished and it was Āryabhaṭa who continued to be universally followed.

Some authorities have thus expressed the view that to meet the popular demand Brahmagupta in the *Khaṇḍakhādyaka* took upon himself the task of simplifying Āryabhata's *ardharātrika* system and in this task he became eminently successful. But it has been supposed that in this task he could not be a mere simplifier or expounder.

प्रायेणार्य्यभटेन व्यवहारः प्रतिदिनं यतोऽशक्यः ।
उद्वाहजातकादिषु तत्समफल लघुतरोक्तिरतः ॥

त्रिभुभिः शनिर्दशशीघ्रं द्वाविंशत्या कुजोऽधिको द्वाभ्याम् ।
चतसृभिरधिको जीवोऽर्द्धरात्रिकार्य्यभट मध्य समाः ॥

—*KK*, I. 1,2,7

1. चैत्रसितादेरुदयाद्भानोर्दिनमासवर्षयुगकल्पाः ।
सृष्ट्यादौ लङ्कायां समं प्रवृत्ताः दिनेऽर्कस्य ॥ —*BrSpSi* I.4

The minor work of Brahmagupta known as *Khaṇḍakhādyaka* has two distinct parts . *Khaṇḍakhādyaka* proper and the *Uttara Khaṇḍakhādyaka* In the first part the astronomical constants are the same as those of Āryabhaṭa's *ardharātrika* system, but the methods of spherical astronomy, calculation of eclipses and other topics are almost the same as in the *Brāhmasphuṭasiddhānta* The corrections for parallax in calculating a solar eclipse is here an important illustration [1]

In the *Uttarakhaṇḍakhādyaka*, Brahmagupta gives corrections to the *Khaṇḍakhādyaka* proper In it are to be found the neat and original methods of interpolation and correction to the longitudes of the aphelia, as also to the dimensions to the epicycles of apsis of the Sun and the Moon [2] while a few additional chapters supply what else is necessary to the first seven chapters of the first part, to make the whole a complete treatise on Indian scientific astronomy It was perhaps through the influence of this supplementary part of the *Khaṇḍakhādyaka* that Brahmagupta's great work, the *Brāhmasphuṭasiddhānta* came to be valued among a distinct school of Indian astronomers For long in this country India, this *Siddhānta* of Brahmagupta has been forming the basis for the calculation of almanacs by astronomers of the orthodox school of Rājasthān, Bombay and others

We might at this stage take up the question · Was Āryabhaṭa the author of two distinct systems of astronomy ? Undoubtedly he was Several authors have written on this subject I may specially mention the name of Prabodha Chandra Sengupta (*Journal of the Department of Letters* Calcutta University, vol XVIII *Bulletin Calcutta Mathematical Society* vol XXII Nos 2 and 3) The reasons advanced by him may be restated in slight details thus In his *Brāhma-*

1 त्र्यागर्भेन विभक्ता दृग्गतिजीवा चतुर्गुणा लम्बनम् ।
लम्बननान्य पञ्चदश गुणितया त्रिज्यया भक्ता ॥
दृक्क्षेपज्या मुक्त्वान्तरा हता लम्बनतिर्भवति ।
स्फुटयोजनकर्णाभ्यां भूव्यासेन च विना स्पष्टे ॥
आर्यभटेनाचार्येण सति लघुनि किमर्थं महत् कृतं कर्म ।
गणितानभिज्ञानात् जाड्यं विजानता यदि तत् सुतराम् ।
—*BrSpSi* XI 23-25 also *KK.* V

2 See UKK. 9

sphuṭasiddhānta. Brahmagupta thus speaks of the two works of Āryabhaṭa

> As in both the works the number of the Sun's revolutions is spoken of as 432,000 years their planetary cycle is clear, i e, of 4 320 000 years Why then is there difference of 300 civil days in the same cycle of the two books ?[1]

Again he says .

> In 14 400 years elapsed of the *Mahāyuga* there is produced a difference of one day in the *audayika* and *ardharātrika* systems[2]

Varāhamihira in the *Pañcasiddhāntikā* writes ·

> Āryabhata maintains that the beginning of the day is to be reckoned from midnight at Lankā, and the *same* teacher again says that the day begins from sunrise at Lankā [3]

Thus from the writings of Brahmagupta and Varāhamihira it is clear that Āryabhaṭa I was the author of both the *audayika* and *ardharātrika* systems of astronomy In Varāhamihira's verse the phrase *sa eva* (स एव) meaning "he undoubtedly" is of special significance It removes the least doubt as to Āryabhaṭa's authorship of both these systems The *audayika* and *ardharātrika* astronomical constants are respectively to be found from the Āryabhatīya and may be deduced from the *Khaṇḍakhādyaka* as well The following is the comparative view of the constants of Varāhamihira and of the present day *Sūrya-siddhānta*

TABLE I

Planetary revolutions in a mahāyuga of 4 320 000 years, according to various authorities

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1 युगरविभगणाः ख्युघृ इति यत् प्रोक्तं तत् तयोर्युगं स्पष्टम् ।
त्रिशती रविदिवसानां तदन्तरं हेतुना केन ॥ —BrSpSi XI 5

2 [illegible] ।
[illegible] ॥ —BrSpSi XI 13

3 लङ्कार्धरात्रसमये दिनप्रवृत्तिं जगाद चार्यभटः ।
भूयः स एव सूर्योदयात् प्रभृत्याह लङ्कायाम् ॥ —PSi XV, 20

Planet	*BrSpSi*	*Āryabhaṭīya*	*Khaṇḍa khādyaka*	Varāha *Sūrya siddhānta* (PSi)	Later or modern *Sūrya siddhānta*
Moon	57 753,300	57 753,336	57 753 336	57 753 336	57 753 336
Sun	4 320 000	4 320 000	4 320 000	4 320 000	3 320 000
Mars	2 296 828 522	2,296 824	2,296 824	2,296 824	2 296 832
Jupiter	364 226 455	364 224	364,220	364 220	364 220
Saturn	146 567 298	146 564	146 564	146 564	145 568
Moon s apogee	—	488 219	488 219	488 219	488,203
Venus	7 022 389 492	7 022,338	7 022 388	7 022 388	7 022,376
Mercury	17 936 998 984	17 937 020	17937 000	17,937 000	17 937 060
Moon s nodes	232 311 168	232 226	232,226	232,226	232,238

TABLE II

Longitudes of the apogees of the orbits of Planets

Planets	*Āryabhaṭīya*	*Khaṇḍa khādyaka*	Varāha *Sūrya siddhānta*	Modern *Surya siddhānta*
Sun	78°	80°	80°	77° 07
Mercury	210°	220°	220°	220° 26′
Venus	90°	80°	80°	79° 49
Mars	118°	110°	110°	130° 00′
Jupiter	180°	160°	160°	171° 16′
Saturn	236°	240°	240°	236° 37

TABLE III

Dimensions of the epicycles of Apsis

Planets	*Āryabhaṭīya*	*Khaṇḍa khādyaka*	Varāha *Sūrya siddhānta*	Modern *Sūrya siddhānta*
Sun	13° 30	14°	14°	13½° 14°
Moon	31° 30	31°	31°	31½° 32°
Mercury	22½ 31½°	28°	28°	28° 30°
Venus	9° 18°	14°	14°	11° 12°
Mars	63°-81°	70°	70°	72° 75°
Jupiter	31½ 36½°	32°	32°	32° 33°
Saturn	40½ 58½	60°	60°	48° 49°

Table IV

Dimensions of the Sighra epicycles (i e conjunctions)

Planet	*Āryabhaṭīya*	*Khaṇḍa-Khadyaka*	Varāha *Sūrya-siddhānta*	Modern *Sūrya-siddhānta*
Saturn	36½°- 40°	40°	40°	39°- 40°
Jupiter	67½°- 72°	72°	72°	70°- 72°
Mars	229½°-239½°	234°	234°	232°-235°
Venus	256½°-265½°	260°	260°	262°-262°
Mercury	130½°-139½°	132°	132°	132°-133°

Table V

Longitudes of the nodes of the orbits of planets

Planets	*Āryabhaṭīya*	*Khaṇḍa-khādyaka siddhanta*	Varāha *Sūrya siddhānta*	Modern *Sūrya-siddhānta*
Saturn	40°	40°	Not stated	Have to be
Jupiter	20°	20°	in the	calculated
Mars	80°	80°	Text	from the
Venus	60°	60°		data of the
Mercury	100°	100°		text

Table VI

Orbital inclinations (geocentric) to the ecliptic

Planets	*Āryabhaṭīya*	*Khaṇḍa-khādyaka*	Varāha *Sūrya-siddhānta*	Modern *Sūrya-siddhānta*
Mars	90′	90′	10′	90′
Mercury	120′	120	135′	120
Jupiter	60	60	101′	60′
Venus	120	120′	101′	120′
Saturn	120	120′	135′	100′

The *Mahābhāskarīya* of Bhāskara I (522 A D.) contains a passage which corroborates the fact that Āryabhaṭa I was the author of both the *audayika* and the *ardharātrika* systems of Indian Astronomy According to Pṛthūdakasvāmin, whose

commentary on the *Brāhmasphuṭasiddhānta* we have the privilege of presenting to the public it is clear that in certain respects Bhāskara and others may be wrong but the Āryabhaṭa s authenticity cannot be questioned Pṛthūdakasvāmin while commenting on the *Brāhmasphuṭasiddhānta* XI 26 writes

> Such a mistake may have been made by Bhāskara and others they have not understood his (Āryabhaṭa s) intention

The passage in the *Mahābhāskarīya* giving constants of the *ardharātrika* system runs as follows (we are giving the translation from Kripa Shankar Shukla s edition on the *Mahābhāskarīya*)

> The astronomical processes which have been set forth above come under the sunrise day reckoning (*audayika* system) In the midnight day reckoning (*ardharātrika* system) too all this is found to occur the difference that exists is being stated (below) [1]

The next fourteen stanzas relate to the midnight day reckoning of Āryabhaṭa I

(i) Civil days and omitted lunar days in a *yuga* and revolution numbers of Mercury and Jupiter are thus given

(To get the corresponding elements of the midnight day-reckoning) add 300 to the number of civil days (in a *yuga*) and subtract the same (number) from the number of omitted lunar days (in a *yuga*) and from the revolution numbers of (the *śīghrocca* of) Mercury and Jupiter subtract 20 and 4 respectively [2]

Thus according to the midnight day reckoning we get

civil days in a *yuga*	=	1 577 917 800
omitted lunar days in a *yuga*	=	25 082 280
revolution number of the *śīghrocca* of Mercury	=	17 937 000
revolution number of Jupiter	=	364 220

1 निरन्ध कमशा प्रोक्तो योऽस्माबौदयिको विधि ।
अ रात्रे ऽत्र सर्वो यो विशेष स कथ्यते । —*MBh VII 21*

2 त्रिशत भूदिने क्षेप्या इ यवमेभ्यो विशोध्यते ।
इ गुर्वोभगणेभ्योऽपि विशतिश्च ततोऽभ्यय ॥ —*MBh VII 22*

(ii) Diameters of the Earth, the Sun and the Moon are thus given :

(In the midnight day-reckoning) the diameter of the Earth is (stated to be) 1,600 *yojanas;* of the Sun 6,480 (*yojanas*) and of the Moon, 480 (*yojanas*)[1].

(iii) Mean distances of the Sun and the Moon are as follows :

The (mean) distance of the Sun is stated to be 689,358 (*yojanas*), *and of the Moon* 51,566 (*yojanas*).[2]

(iv) Longitudes of the apogees of the planets are as follows: 160, 80, 240, 110, and 220 are in degrees the longitudes of the apogees of Jupiter, Venus, Saturn, Mars and Mercury respectively.[3]

(v) *Manda* and *Śighra* epicycles of the planets are as follows :

The *Manda* epicycles (of the same planets) are 32, 14, 60, 70, and 28 (degrees) respectively; and the *Śighra* epicycles are 72, 260, 40, 234, and 132 (degrees) respectively. The Sun's apogee and epicycle are the same as those of Venus (i. e. 80° and 14° respectively).

The Moon's epicycle in the midnight day-reckoning is stated to be 31 (degrees).[4]

(vi) The positions of the so called *manda* and *śighra pātas* of the planets are given below:

(The following directions for) the degrees of the (*manda* and *śighra*) *pātas* of the planets as devised

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1. अष्टिश्शतगुणा व्यासो योजनानां भुवो रवेः ।
खाष्टाब्ध्यङ्गानि शीतांशोः शून्यवस्वब्धयस्तथा ॥ —23

2. वरिवन्द्रिय गुणच्छिद्रवस्वङ्गानि विभावसोः ।
अङ्गाङ्गेष्वेक भूतानि चन्द्रकर्णः प्रकीर्तितः ॥ —24

3. अष्टिरष्टौ जिनाहद्रा विंशतिर्द्व्यधिकाः क्रमात् ।
दशघ्ना गुरुशुक्रार्कि भौमज्ञांशाः स्वमन्दजाः ॥ —25

4. मन्दवृत्तानि द्वात्रिंशन्मनवः षष्टिरेव च ।
खाद्रयो वसुदस्राः स्युः शीघ्रवृत्तान्यथ क्रमात् ॥ —26
द्व्यद्रयः खाङ्गनेत्राणि खाब्धयोऽब्ध्यग्निदस्रकाः ।
द्व्यग्नीन्दवो रवेर्मन्दं शुक्रवद् वृत्तमेव च ॥ —27
एकत्रिंशत्तपामर्तुः अर्धरात्रे विधीयते । —*MBh. VII 26-28 (i)*

> (under the midnight day reckoning) should be noted carefully by learned scholars
>
> Add 180° to the longitudes of the *mandoccas* (apogees) and *śighroccas* of Mercury and Venus and subtract 3 signs from the *mandoccas* (apogees) and *śighroccas* of the remaining planets Then are obtained the longitudes of the *manda* and *śighra pātas* of the planets (Also) add 2 degrees to the longitudes of the *manda pātas* and *śighroccas* of Venus Saturn and Jupiter, and 1½ degrees to those of Mars and Mercury (It should be noted that) the *śighra pātas* have been stated for all the planets excepting Mercury (Mercury does not have a *śighra pāta*) [1]

That is to say the longitudes of the *manda pātas* of Mars, Mercury Jupiter Venus and Saturn are 21.5, 41.5, 72, 262 and 152 degrees respectively and the longitudes of the *śighra pātas* of Mars Jupiter Venus and Saturn are (*śighrocca*−88.5°) (*śighrocca* −88°), (*śighrocca*+182°) and (*śighrocca*−88°) respectively

(vii) A rule for finding the *celestial latitude of a planet* is as follows

> (From the longitude of a planet severally) subtract the longitudes of its (*manda* and *śighra*) *pātas* and therefrom calculate (as usual) the corresponding celestial latitude of that planet Add them or take their difference according as they are of like or unlike directions Then is obtained the true celestial latitude of that particular planet The true celestial latitude of any other planet is also obtained in the same way The remaining (astronomical) determinations are the same as stated before. This all is in brief the difference of the other *tantra* (embodying the midnight day-reckon-

the nodes of the planets to be the same as in the *Āryabhaṭīya*. All these are the same as given by Brahmagupta in the *Khaṇḍakhādyaka*.

In the stanza 33 we have rules for finding the geocentric longitudes of planets which may be taken to be the same as in the *Khaṇḍakhādyaka*[1]; compare these values with those in the *Sūryasiddhānta* of Varāhamihira in the Pañcasiddhāntikā, XVII. 6, but slight different from the *Āryabhaṭīya*.[2]

The last stanza of the *Mahābhāskarīya* (35) gives the dimensions in *yojanas* of the orbits of planets ; these are the same as in the modern *Sūryasiddhānta*.

Thus we find a great semblance in the constants as given by the *Mahābhāskarīya* of Bhāskara I, of the Sūryasiddhānta as given by the *Pañcasiddhāntikā* and understood by Varāhamihira, and also the constants as given by Brahmagupta in his treatises, specially the *Khaṇḍakhādyaka*. It must not be forgotten that the same Āryabhata I who is the celeberated author of the *Āryabhaṭīya* is also the author of another treatise very often referred to as the *Tantra*

I shall quote Prabodha Chandra Sengupta in connection with these similarities, and the great influence of Āryabhata on Indian Astronomy. He writes in his Introduction to the *Khaṇḍakhādyaka* as follows :

> We have shown that there is much resemblance in the constants between the *Sūryasiddhānta* of Varāha and the *Khaṇḍakhādyaka* and for the matter of that with the *Tantrāntara* of Āryabhaṭa I. In my papers "Āryabhata and Āryabhaṭa's Lost Work", I have established the fact that the *Sūryasiddhānta* as it existed before the time of Varāha, was made more accurate by him by borrowing the constants from Āryabhaṭa's *ardharātrika* system That there was a *Sūryasiddhānta*

१. शीघ्रफलार्द्धं मध्ये मन्दफलार्द्धं च मन्दशीघ्रफले ।
सकले मध्ये स्पष्टः शीघ्रं मध्योनित केन्द्रम् ॥ —*KK. II. 18*

२. मन्दोच्चाच्छीघ्रोच्चादर्धमृणधनं ग्रहेषु मन्देषु ।
मन्दोच्चात्स्फुट मध्याश्शीघ्रोच्चाच्च स्फुटा ज्ञेयाः ॥
शीघ्रोच्चादर्धोन कर्तव्यमृण धनं स्वमन्दोच्चे ।
स्फुटमध्यौ तु भृगुबुधौ सिद्धान्मन्दात्स्फुटौ भवतः ॥ —*Arya. III. 23-24*

before the time of Varāha, is seen from Section 6 of the Table on page xii given before This point is made clear from another consideration viz, the star table in the modern *Sūryasiddhānta*, which unmistakably points to the conclusion that the longitudes of some stars, e g, Spica *etc*, correspond to a time much anterior to that of Āryabhaṭa I The great fame of Ārya bhaṭa I induced Varāha the first maker of a *neo Surya siddhānta* to use the elements of Āryabhaṭa s *ardha rātrika* system to supplant the older materials in it No wonder therefore that there is an opinion in favour of the hypothesis that Āryabhaṭa I was the author of the *Sūryasiddhānta* If there were a shadow of truth in it Varāha would have admitted it Albe runi indeed says that the *Sūryasiddhānta* was compos ed by *Lāṭa* (Alberuni s India translated by Sachau Vol I p 153) We now know that this Lāṭa or Lāṭa deva was one of the first pupils of Āryabhaṭa I He was the expounder of the Romaka and Paulīśa *Siddhāntas* as we learn from Varāhamihira s *Pañcasiddhāntikā* (I 3) As Alberuni s statement is not corroborated by Varāha, we are not inclined to take it as correct None of the earlier writers suggest that the *Sūryasiddhānta* was in any way modified or changed by Ārya bhaṭa I

It has now been established beyond doubt that the same Āryabhaṭa was the author of the *Āryabhaṭīya* and another *Tantra* which is now lost There is reason in support of hypothesis that this *Tantra* itself was the first work of Āryabhaṭa I and that the *Āryabhaṭīya* was the second work from the order in which Varāha mentions them in the Stanza quoted earlier If this hypothesis be true the stanza in the *Āryabhaṭīya*[1]

which was translated by me as

"Now when sixty times sixty years and three quarter *yugas* also have elapsed twenty increased by three years have elapsed since my birth

1 षष्ट्यब्दानां षष्टिर्यदा व्यतीतास्त्रयश्च युगपादाः ।
त्र्यधिका विंशतिरब्दास्तदेह मम जन्मनोऽतीताः ॥ —*Ārya III 10*

should now be *translated thus* :

"In this *Mahāyuga* when sixty times sixty years and three quarter yugas also had passed, twenty increased by three years had elapsed since my birth."

Now Bhāskara I the author of the *Mahābhāskarīya* and the *Lagubhāskarīya*, wrote a commentary on the *Āryabhaṭīya*. The author commenting on this stanza observes that :

> "Or this was addressed by Āryabhaṭa when expounding the science to Pāṇḍuraṅgasvāmin, Lāṭadeva *Niḥśanku and other pupils*."[1]

This direct pupil of Āryabhaṭa I also says that this stanza does not show that the *Āryabhaṭīya* was composed when Āryabhata *I was only 23 years old but refers to the time when he* probably began his career as a teacher of Astronomy.

Senagupta out of his discussion concludes that we are not justified *in accepting that the Āryabhaṭīya was composed* when Āryabhaṭa was only 23 years of age. This treatise as it exists in the present form must have been the composition of a mature age; it is a treatise highly finished in form; the date mentioned in this great work refers to a date when its author became a *reputed guru* or *teacher*.

Alberuni and Brahmagupta

Dr. E.C. Sachau in his translation of Alberuni's India (vol. II. p. 304) speaks of Brahmagupta in *the following words* :

> Brahmagupta holds a remarkable place in the history of Eastern civilization. It was he who taught the Arabs astronomy *before they became acquainted* with Ptolemy; for the famous *Sindhind* of Arabian literature, frequently mentioned but not yet brought to light, is a translation of his *Brāhmasiddhānta* ; and the only other book on Indian astronomy, called Atarkand, which they knew, was a translation of his *Khaṇḍakhādyaka*.

Brahmagupta, the celebrated author of the *Brāhmasphuṭasiddhānta*, has another great work as we have said before to his credit which goes by the name *Khaṇḍakhādyaka*. This has

1. एतदेवाचार्य्यार्य्यभटस्य शास्त्रव्याख्यान समये वा पाण्डुरङ्गस्वामिलाटदेवनिःशङ्कुप्रभृतिभ्यः प्रोवाच ।

already been said that perhaps to meet the popular demand, Brahmagupta in this treatise took upon himself the task of simplifying Āryabhaṭa's *ardharātrika* system or the system of midnight day reckoning Alberuni the author of the *Indika* has made several references or quotations from the *Khaṇḍakhādyaka* proper and also its supplement, known as *Uttara-Khaṇḍakhādyaka*

(a) There is a reference to the accepted circumference of the Earth, as given in the *Khaṇḍakhādyaka* (Sachau s *Alberuni*, Vol I p 312)

Multiply the difference in longitude (from Ujjayinī) by the (mean) daily motion of a planet (in minutes) and divide by 4 800 apply the quotient taken as minutes negatively in places east of the meridian line of Ujjayinī and positively in places lying west[1]

(b) The rules for finding the *ahargana* as given in the *Khaṇḍakhādyaka* in I 3 5 (Sachau s Alberuni Vol II 46-47) to which Dr Schram adds a valuable annotation the constants being taken from the later Pauliśa Tantra as known to Bhattotpala This Pauliśa astronomy is derived from Āryabhaṭa I s *ardharātrika* system[2]

(c) A quotation from the Uttara *Khaṇḍakhādyaka* (Sachau's Alberuni Vol II pp 84-86) which Sengupta has given in his translation Chapter X pp 148 152

(d) A quotation also probably from the *Uttara Khaṇḍakhādyaka* (Sachau s Alberuni Vol II p 87) These stanzas are found in the *Brāhmasphuṭasiddhānta*, XIV, 47 52 also quoted by Bhaṭṭotpala as occurring in the *Brahma Siddhānta* in his commentary on the Bṛhat Saṁhitā, IV 7 The manuscripts which Sengupta used did not show them as occurring in the *Uttara Khaṇḍakhādyaka* These relate to the dimensions of the *nakṣatras* as seen as distinguished from the same as calculated

1 उज्जयिनी-याम्योत्तर-रेखाया प्राग्रुणं धनं पश्चात् ।
देशान्तर भुक्तिवधात् ख खाष्टवेदैः कलाद्याप्तम् ॥
—KK I, 15

2 —KK Pt B Misra s edition p 145

(e) Two quotations from the *Uttara Khaṇḍakhādyaka* relating to the celestial co-ordinates of Canopus and Sirius (Sachau's Alberuni Vol II p 91) Present manuscripts do not show these stanzas which are probably the same as stanzas 35-36 and 40 of Chapter X of the *Brāhmasphuṭasiddhānta*

(f) Two quotations from the *Khaṇḍakhādyaka* proper as alleged by Alberuni (Sachau's Alberuni Vol II p 116) According to Āmarāja the first is a couple of stanzas of which the author is Bhaṭṭotpala and not Brahmagupta The second quotation cannot be traced These relate to finding the possibility of an eclipse whether of the Sun or of the Moon

(g) Two quotations from the *Khaṇḍakhādyaka* proper as asserted by Alberuni (Sachau's Alberuni Vol II p 119) These relate to finding the Lords of the year and of the month According to Āmarāja the rules in question were given by Bhaṭṭotpala and not by Brahmagupta Pṛthūdaka in his commentary on the first chapter at its concluding portion says

> In this work the *Khaṇḍakhādyaka* the teacher (Brahmagupta) has not given the rules for finding the Lords of the year and the month[1]

1 अथाऽत्र खण्डखाद्यके वर्षाधिपमासाधिपौ नयनमाचार्येण नाभिहितम् ।

— o —

Reference

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Chapter IV

Brahmagupta's Originality in the Khaṇḍakhādyaka

Sengupta in his Indroduction to the Commentary of the *Khandakhadyaka* has discussed this point We shall reproduce here some of the points mentioned by him

Brahmagupta's Khandakhadyaka

(i) Brahmagupta does not accept the system of Āryabhaṭa but has simplified it in the *Khaṇḍakhādyaka* proper, and here he has given the system which he thinks to be correct

Uttar Khanda Khādyaka

(ii) In the *Uttara-Khaṇḍakhādyaka*, he has further corrected some of his results, given earlier in the *Khandakhādyaka* proper In the proper *Khaṇḍakhādyaka* Brahmagupta assignes to the longitude of the Sun's apogee the value 80°, whereas in the *Uttara* text he corrects it to 77° (UKK 4)

> As the process of finding the apparent places of planets as given by Āryabhaṭa does not make them agree with observation, I shall, therefore, speak of this process Of the Sun the apogee is at two signs and seventeen degrees (2 signs 17°=60 plus 17 degrees=77°) [1]

Compare this with the value given in the *Khaṇḍakhādyaka* proper (I 13)[2]

The longitude of the Sun s apogee is 80° [KK I,13] (The Sun s apogee is 80° or two signs plus 20 degrees) *inocco* means

1 [illegible] ।
[illegible] ॥ *UKK. IX 4*

2 [illegible] ।
[illegible] ॥ *KK I, 13*

mandocca of the Sun). The value given in the *Pañcasiddhāntikā*, IX.7-8) is also the same.

Let us compare it with the present value. According to the astronomical constants as given in the *Conn. des Temps.*, the longitude of the Sun's apogee in 499 A.D. (i.e. 1,400 years before 1900 A.D.) was

—77° 19′19.44″ according to *Conn.des Temps*' equation.

—76° 40′37.22″ according to Newcomb's equation.

The mean of these two values is very nearly 77° as given by Brahmagupta in the *Uttara* text. Thus the value given by Brahmagupta is more correct than the value given by Āryabhaṭa. The *Āryabhaṭīya* gives the value 78° which is less correct.

Brahmagupta more correct than Āryabhaṭa

(iii) Brahmagupta detected that Āryabhata had made the Moon's apogee quicker and nodes slower, than they really are. In both the cases, Brahmagupta made rather an over-correction. We shall give the extract from *Uttara-Khaṇḍakhādyaka* in this connection :

> Multiply the *ahargaṇa* by 110, increase the product by 511 and divide by 30, 31; subtract the result taken as revolutions, etc., from the mean Moon; the final result is the Moon's apogee.[1]

Evidently Brahmagupta assumes that the anomalistic month=3031/110 days. This convergent to the anomalistic month was known to the author of the *Vasiṣṭha Siddhānta* as summarised in the *Pañcasiddhāntikā*[2] (II-2-6).

According to Brahmagupta, the length of the anomalistic month

$= \frac{1582236450000-4320000000}{57753300000-488105858}$ days. (*BrSpSi*. I 15,16,18. and 20)

= 27.55454641 days which is for 1900 A. D.

= 27.5545502 days according to Radau.

= 27.554602 according to the *Āryabhatīya*.

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1. द्युगणात् खरुद्रगुणिताद् एकशरयुक्ताच्छशिद्विरामनिहृतात् ।
भगणादि फलं शोध्यं चन्द्राच्छशाङ्कोच्चम् ॥ (*UKK*. IX, 5)

Here also Brahmagupta is more accurate.

Again, the length of the sidereal period of the Moon's apogee

$$= \frac{1577918450000}{488105858} \text{ days}$$

$$= 3232.732048 \text{ days.}$$

Āryabhaṭa's value of the same is 3231.987844 days, and the modern value is 3232.3754 days. Hence Brahmagupta's result is by 0.3566 of a day out, while Āryabhaṭa's is by 0.3876 of a day in.

> Further in the *Uttara-Khaṇḍakhādyaka*(IX.10) we have. Deduct 3541 from the *ahargana*, divide the remainder by 6792; subtract the quotient that is obtained in revolutions etc. from the circle: the result is the longitude of the ascending node.[1] (IX.10)

Here Brahmagupta gives the approximate period of the sidereal revolution of the Moon's node to be=6792 days. This according to his *Brāhmasphuṭasiddhānta* $= \frac{1577916450000}{232311168}$ days

= 6792.25396 days, which according to Lockyer would be 6793.39108 days and according to the *Khaṇḍakhādyaka* proper is 6794.75083 days. Hence Brahmagupta's attempt to correction makes the node quicker than it actually is.

Brahmagupta corrects Mars's Aphelion Point

(iv) Again Brahmagupta states that the longitude of Mars's aphelion should be increased by 17° and that of Jupiter by 10°. Evidently here too, Brahmagupta is more correct than Āryabhaṭa. The passage in the *Uttara Khaṇḍakhādyaka* is as follows in this connection:

> Of Mars the apogee (the aphelion point) is to be increased by 17°, that of Jupiter by 10°, from the *sighra* of Venus 74 are to be subtracted. Saturn's equation of apsis should be decreased by its one-fifth; the *sighra* equation of Mercury should be increased by

one-sixteenth [1]

This stanza says that in 499 A D Mars s aphelion point had a longitude of 127° of Jupiter the longitude of the aphelion was 170° (*KK.II 6*[2])

According to Newcomb s rule the longitude of the aphelion point of Mars in 499 A D works out to have been = 128°28 12 According to the *Conn des Temps* rule the same was 128°27 51 Hence Brahmagupta s determination of Mars s aphelion is correct within 1°30 and is therefore quite satisfactory According to the *Khaṇḍakhādyaka* proper it was 110° and according to the Āryabhaṭīya 118°

Of planets beginning with Mars the degrees of longitude of the apogees are respectively 11 22, 16 8 and 24 each multiplied by 10 (*KK .II 6*)

Thus the longitudes of apogees of Mars = 110 (3 signs 20°) of Mercury = 220° (7 signs 10°) of Jupiter = 160° (5 signs 10°) of Venus = 80° (2 signs 20°) and of Saturn = 240° (8 signs) Compare these values with those given in the *Pañcasiddhāntika* XVII 2 (the *Sūrya sidahānta)*

Again according to this stanza Jupiter s aphelion had a longitude of 170° in 499 A D

According to *Conn. des Temps* rule the same was 170°25 Thus here too Brahmagupta is very accurate According to the *Khaṇḍakhādyaka* proper Jupiter s aphelion had a longitude of 160° (*KK II.6*) and according to the *Āryabhaṭīya* the value was 160°

Brahmagupta First to Use Second Differences

All these illustrations reproduced here very well establish the point that the great Indian astronomers from Āryabhaṭa I to Brahmagupta were aware of the methods of separating the two distinct planetary inequalities viz that of the apsis and of conjunction in the cases of the five star planets (*PS*: Introduction Lii) In the *Khaṇḍakhādyaka* Brahmagupta having given the 'sines and the equations of the Sun and the Moon

1 [illegible] ।
[illegible] ॥
[illegible] ॥ —*UKK IX 11*

2 [illegible] । —*KK II 6*

at the interval of 15° of arc of the mean anomaly, in the *Uttara Khandakhādyaka* teaches, for the first time in the history of mathematics, the improved rules for interpolation by using the *second difference*. This very important feature I am reproducing here from the translation by Sengupta of the verse [1] (*UKK*. 8).

> Multiply the residual arc left after division by 900′ (i.e. by 15°), by half the difference of the tabular difference passed over and that to be passed over and divide by 900 (i.e. 15°): by the result increase or decrease, as the case may be, half the sum of the same two tabular differences; the result which, whether less or greater than the tabular difference to be passed, is the true tabular difference to be passed over (*UKK*. 8)

The rule given here applies to the case of all functions hitherto considered in the *Khandakhādyaka*, which are tabulated at the difference of 15° of arc of the argument. They are:

(i) the tabular differences of the Sun's equation

(ii) the tabular differences of the Moon's equation

(iii) the tabular differences of the 'sines'.

Sengupta has illustrated the rule by an example belonging to the table of sines.

Illustration—To find the 'sine' of 57°

Brahmagupta's table of sines in the *Khandakhādyaka* is as follows:

> Thirty increased severally by nine, six and one, twenty-four, fifteen and five are the tabular differences of sines at intervals of half a sign. For any arc the 'sine' is the sum of the parts passed over, increased by the proportional part of the tabular difference to be passed over [2] (*KK* I.30, also III 6)

This can be shown in the tabular form thus

Arc	'Sine'	Tabular difference	Second difference
0°	0		
15°	39	39	
30°	75	36	−3
45°	106	31	−5
60°	130	24	−7
75°	145	15	−9
90°	150	5	−10

Now 57° = 3420 minutes $= 900 \times 3 + 720$ Thus three of the tabular differences are considered as passed over, the last one being 31 and the one to be passed over is 24

The true tabular difference by the rule for arc 57°

$$= \frac{31+24}{2} - \frac{720}{900} \times \frac{31-24}{2}$$

Hence the 'sine' of 57°

$$= 39+36+31+\frac{720}{900}\left\{\frac{31+24}{2} - \frac{720}{900} \times \frac{31-24}{2}\right\}$$

$$= 125\ 76$$

As worked out from the logarithm tables the same comes out to be 125 80

Again 'sine of 57° from Brahmagupta's formula

$$= 106+\frac{720}{900} \times 24 + \frac{31-24}{2} \times \frac{720}{900} - \left(\frac{720}{900}\right) \times \frac{31-24}{2}$$

$$= 106+\frac{720}{900} \times 24 + \frac{720}{900}\left\{\frac{720}{900} - 1\right\} \times \frac{24-31}{2}$$

This in fact is the modern form the interpolation equation up to the term containing the second difference *Brahmagupta thus* takes a decidedly improved step here and is undoubtedly the *first man in the history of mathematics* who has done this One should also remember that *in the case where the function is not* tabulated at a constant interval Brahmagupta s rule is remarkable

Brahmagupta First to Introduce Sine Rule in Indian Plane Trigonometry

In this connection, we shall reproduce the following verse from the *Khaṇḍakhādyaka* :

> Multiply the 'sine' of the *(Śighra)* anomaly by the 'sine' of the maximum Śighra equation and divide by the 'sine' of the corresponding *Śighra* equation, the result is the '*Śighra* hypotenuse' when the *(Śighra)* anomaly is half a circle, this śighra hypotenuse is equal to the radius diminished by the 'sine' of the maximum equation ; when the anomaly is equal to the whole circle, the same is equal to the radius increased by the same 'sine' of the maximum equation.[1]

$$\text{Now } \log \left\{\frac{126}{594}\right\} = \bar{1}.3265841$$

The values of the ∠PMK and the ∠PEM and Brahmagupta s values as given in the verse[1] are presented below in a tabular form

∠PMK=	28°	60°	90°	121°	135°	148°	164°	173°
∠PEM=	10°58	23°1′	33°1′	39°56′	40°23	37°31′	25°32	12°35
Brahmagupta s ∠PEM=	11°	23°	33°	40°	40°30	37°30′	25°30	12°30′

It will be seen that Brahmagupta gives the values of the equation within 1/8th of a degree It seems inexplicable why such discrepancies should remain in Brahmagupta s calculations It is probable that he wanted to state his equations to the nearest half a degree

Now we shall take up the *Sighra equations of Mars* and then revert to the Sine Rule We have in the *Khaṇḍakhādyaka*

Mars, by the degrees of *Śighra* anomaly (i e anomaly of conjunction) of 28 *getting at the corresponding equation of 11° rises (heliacally) in the east* by the next 32° gets 12° more of the equation , by the next 30° 10° more , by the next 31°,7°, more by next 14°, half a degree , these are positive by the next 13°, negative 3° , by the next 16° , negative 12° after this he is retrograde , by the next 9°, negative 13° , by the next 7°, negative 12½° After this the parts of the equations occur in the reverse order[1]

On the basis of this we have the *following table of the Śighra* equations for Mars ,

Degrees of anomaly of conjunction	Equation of conjunction	Phenomena
0°	0°	Motion direct
28°	+11°	*Rises in the east*

1 भौमोऽष्टयमे रुद्रान् भुक्त्वा पूर्वोदितोऽर्कैरन्य ।
खगुणैरंशाऽरूपगुणे सप्तांशा मनुभिरर्धं शरान् ॥
धनमृणमग्नि शराङ्कैर्यमनिरश्च भाग्य वक्री ।
नवभिस्त्रयोदशनगैर्सार्धार्कान् विलोमोऽत ॥ *KK I* 6-9

Degrees of anomaly of conjunction	Equation of conjunction	Phenomena
60°	11+12=+23°	
90°	23+10=+33°	
121°	33+ 7=+40°	
135°	40+ ½=+40°30	
148°	40°30 −3°=+37° 30	
164°	37°30 −12°=+25°30	Retrograde motion begins.
173°	25°30 −13°=+12°30	
180°	12°30−12°30 =0°0′	
187°	−12°30	
196°	−25°30	Direct motion begins.
212°	−37°30	
225°	−40°30	
239°	−40°	
270°	−33°	
300°	−23°	
332°	−11°	Sets in the west
360°	0°	

Now we come back to our discussion on the verse VI I

The *Śighra* hypotenuse spoken of here is EP, when SP or EM is taken to be R when ∠PEM is a maximum PM is its 'sine

It would be seen from the figure that

$$EP=\frac{R \sin PMK \times PM}{R\sin PEM}$$

This may again be written as

$$\frac{EP}{\sin PMK}=\frac{PM}{\sin PEM}$$

This is equivalent to the sine rule for a triangle in plane trigonometry Brahmagupta is here seen to be the first person to give it in Indian mathematics This expression reminds us of the famous relationship in respect to triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Brahmagupta corrects
Dimensions of the Epicycle of Apsis

Brahmagupta corrects the dimensions of the epicycles of apsis of the Sun and Moon by—1/42 nd part and 1/48th parts respectively. The reference may be made to the following verse in the *Uttara Khaṇḍakhādyaka :*

> The Sun's equations are to be made less by *dvikṛtāṁśonam* (1/42nd) and the Moon's equations, increased by *vasuvedabhāgayutam* (1/48th). Multiply the Sun's equation a planet's daily motion in minutes and divide by the number of minutes of a whole circle and this is called *Bhujāntara* correction and applied in the same way to the planet as the equation is applied to the Sun.[1]

The Sun's epicycle of apsis has the dimension 14° in the *Khaṇḍakhādyaka* proper. With the correction introduced here, the value becomes $14^\circ \left\{1-\frac{1}{42}\right\} = 13^\circ\ 40'$.

The correction to the Moon's equations would make the epicycle's dimension changed from 31° to $31^\circ \left\{1+\frac{1}{48}\right\} = 31^\circ 38' 45''$.

Pṛthūdaka's commentary further corrects it to $31^\circ(1+\frac{1}{52}) = 31^\circ 35'$

Brahmagupta 's correction to Saturn's epicycle of apsis is—1/5th part and that to the *Śighra epicycle of Mercury 1/16th* part as seen in the verse :

> Of Mars the apogee (the aphelion point) is to be increased by 17°, that of Jupiter by 10° ; from the *Śighra* of Venus 74' are to be subtracted ; Saturn's equation of apsis should be decreased by its one-fifth and the *Śighra* equation of Mercury should be increased by one-sixteenth.[2]

In the *Khaṇḍakhādyaka* proper (II.6), we have been given the longitudes of the apogee of planets : Mars 11°, Mercury 22°, Jupiter 160°, Venus 80° and Saturn 240° Now with these corrections introduced in the *Uttara Khaṇḍakhādyaka* in the above

1. द्विकृतांशोनं रविफलमिन्दोर्वसुवेदभागयुतम् ।
   [illegible] भुजान्तरं रविवत् ॥ *UKK. IX. 9*
2. [illegible] ।
   [illegible] ।
   [illegible] ॥ *UKK.IX.11*

verse, the aphelion point of Mars in 499 A D had a longitude of 110 plus 17=127°, of Jupiter 160 plus 10=170°

According to Newcomb s rule, the longitude of the aphelion point of Mars in 499 A D works out to have been 128°28 12 The same according to *Conn des Temps* rule would be 128°27 51" Hence Brahmgupta s determination of Mars s aphelion is correct within 1°30 , and may be, therefore regarded as very satisfactory According to the *Khandakhādyaka* proper this value was, as already said, 110°, while according to the *Āryabhaṭīya* it was 118°

The same may be said regarding Jupiter s aphelion According to the *Khaṇḍakhādyaka* proper the value of its longitude is *160°, according to the Āryabhaṭīya it was 180°*, according to *Conn des Temps* rule, it would be in 499A D 170°25 and the value given by Brahmagupta in the *Uttara Khaṇḍakhādyaka*, it *is 170°*

We have thus shown by many illustrations the important corrections introduced by Brahmagupta in his *Khaṇḍakhādyaka* specially the *Uttara* part Brahmagupta was highly original in his methods of calculations, accuracies and interpolations He introduced new ideas in mathematics He went much ahead Āryabhaṭa in many details He so many times did not follow Āryabhaṭa in calculations In the *Khaṇḍakhādyaka* proper, his treatment of parallax in the calculation of solar eclipses is different from that of Āryabhaṭa The methods followed here are the same as propounded by him in the *Brāhmasphuṭasiddhānta*[1]

Senagupta is right when he says As has already been remarked these corrections and innovations in the *Uttara Khaṇḍakhādyaka* paved the way for the acceptance of his great work the *Brāhmasphuṭasiddhānta* as a standard work on astronomy by the western Indian school of astronomers The directness of the treatment of topics and the simplicity of calculations taught in the *Khaṇḍakhādyaka* made it very neat handbook for the beginner These two works of Brahmagupta were perhaps the only astronomical works in circulations in western India when the Arabs conquered Sind early in the eighth century

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1 *On Parallax*—
नाग्नै चतुष्कविधिना सर्वत्र समो यतस्तत स्थूल ।
मानार्थं कर्म मदर्थं फलमात्रमतेन लघुनि सति ॥ *BrSpS*ı XI 33

(712 A.D.) and the new conquerers learnt Indian astronomy and mathematics from these works as has been observed by Sachau. Alberuni who came to India early in the 11th century of the Christian era, learnt Indian astronomy chiefly by studying the *Khandakhādyaka* and the *Bṛhat Saṁhitā* of Varāhamihira, and both of them with the help of commentary of Bhattotpala.

—:o:—

Reference

P.C. Sengupta : *The Khandakhādyaka, 1934.*

CHAPTER VI

Indian Luni-Solar Astronomy

In this chapter, it is proposed to give an account of astronomical constants and the equations in Indian luni-solar astronomy and to present a comparative view of these quantities with the corresponding ones in Greek and modern Astronomy. This account has been reproduced from P C Sengupta's Appendix I of the *Khandakhādyaka*. It has been shown that in many cases the Indian values of these constants are more accurate than the Greek values and in Indian lunar astronomy the equations or inequalities discovered are the most startling.

Solar Astronomy

In *solar astronomy* the length of the year was determined by Āryabhaṭa[1] from the heliacal risings of some bright star at the intervals of 365 and 366 days.

(1) The year according to the Āryabhaṭīya

$$= \frac{1577917500}{4320000} \text{ days} = 365.2586805 \text{ days}$$

$= 365$ da 6 hrs 12 mins 29 64 secs

(2) The same

$$= \frac{1577917800}{4320000} \text{ days} = 365.25875 \text{ days}$$

$= 365$ da 6 hrs 12 mins 36 secs, according to the *Khaṇḍakhādyaka*, the *Sūryasiddhānta* of Varāha and the modern *Sūryasiddhānta*

(3) It is

$$= \frac{1577916450}{4320000} \text{ days} = 365.2584375 \text{ days}$$

$= 365$ da 6 hrs 12 mins 9 secs, according to the

1 P C Sengupta Āryabhaṭa's Method of determining the Mean Motions of Planets *Bulletin of the Calcutta Mathematical Society* Vol. XII, No. 3

Brāhmasphuta Siddhānta of Brahmagupta.

Now the mean sidereal year

=365 da 6 hrs. 9 mins. 9 3 secs. (Lockyer).

The mean anomalistic year

=365 da. 6 hrs. 13 mins. 49.3 secs. (Lockyer).

The mean tropical year

=365 da. 5 hrs. 48 mins. 46 054 secs (Lockyer).

Though we take that Indian year was designed to be the sidereal year, it approached most closely the anomalistic year; and its excess over the sidereal year was about 3 minutes From this consideration it appears that the Indian astronomers were justified in taking the Sun's apogee to be fixed.

Against the error of+3 min. in the Indian sidereal year, we may point out that—

(1) The Hipparchus Ptolemy tropical year

=365 da. 14' 48" in sexagesimal units.[1]

=365 da 5 hrs 55 min. 12 secs, which has an error of about+6 min.

(2) Meton's sidereal year

$$= \left\{365+\tfrac{1}{4}+\frac{1}{76}\right\} \text{ days}^{2}$$

=365 da. 6 hrs. 18 min. 57 secs. which has an error of +9 min. 48 secs. nearly.

(3) The Babylonian sidereal-year was 4½ min. too long.[3]

Thus the Indian value of it is closer to the true value.

Again in 150 A D. the longitude of the Sun's apogee according to the *Conn des Temps* was

$$=101^\circ\, 13'\, 15''.17\text{-}6189''.03 \left\{\frac{1900-150}{100}\right\}$$

$$=1''.63 \times \left\{\frac{1900-150}{100}\right\}^2$$

=71° 16' 26". 37

while Ptolemy states it to be 65° 30' which was wrong by—5° 36' 27":

1. *Syntaxis*, Karl Manitius's edition, Vol I. p 146.
2. Ibid, p 145
3. *Encyclopaedia Britannica*, History of Astronomy.
4. *Syntaxis*, Vol I. p 143. The Romaka Siddhānta of the *Pañca-siddhāntikā*, VIII 2, indicates the Sun's apogee to be at longitude of 75°, this was perhaps a correction made by Lāṭadeva to the Greek constant.

In 500 A D (Āryabhaṭa s time) the longitude of the Sun's apogee by the same rule works out to be=77°19 19 44".

Āryabhata states this to be 78° in the Āryabhaṭiya, Brahmagupta in the Uttarādhyāya of the *Khaṇḍakhādyaka* states the same to be 77°, while the Khaṇḍakhādyaka gives it as=80° Hence the Indian findings of the longitude of the Sun's apogee were more accurate

Again as to the Sun s equations of the centre we find that the *Āryabhaṭīya* states the periphery of the Sun's epicycle to be 13°30. The *Khaṇḍakhādyaka* gives it as 14°, while according to the Indian form Ptolemy's value of the same is 15° Hence according to these writers, the Sun's equations at 90° of the mean anomaly are :—

According to the *Āryabhaṭīya*			=2° 8' 54".
"	"	*Khaṇḍakhādyaka*	=2° 1' 40"
"	"	Brahmagupta	=2° 7 20"
"	"	Ptolemy	=2° 23 3"
The modern value			=1° 55' 97"

Thus the Indian equations of the Sun are in general by more correct than the Greek ones The Indian constants in solar astronomy are thus, generally more accurate than the Greek ones We now turn to the Indian Lunar astronomy

Lunar Astronomy

Before discussing the constants in Indian lunar astronomy it is necessary to state something as to the time when the Moon was observed by our ancient astronomers and the astronomers from Āryabhaṭa I (499 A D to Pṛthūdaka Svāmi (864 A.D) The months were reckoned from the first visibility of the crescent at the time of the *Mahābhārata* (1400 B C) We have a passage in the Bhismaparva where Vyāsa speaks of the evil omens on the eve of the Kurukṣetra war thus—

चन्द्रसूर्यावुभौ ग्रस्तावेकमासीं त्रयोदशीम् ।

'That the Moon and the Sun have been both eclipsed on the 13th days of the light and dark halves of the same month "

The eclipses could not take place on the 13th days of the month unless the months were reckoned from the first visibility of the crescent This was the custom in Babylonia and it has still survived in the Mahomedan world Even in the *Pañca siddhāntikā* of Varāhamihira (540 A D), there is a special

chapter on शशि-दर्शनम् or the first visibility of the crescent. It is thus clear that the practice was to observe the Moon when very near the Sun.

Again Āryabhata says that 'रवीन्दुयोगात् प्रसाधितश्चेन्दुः', "the Moon was determined from her conjunctions with the Sun." The Moon was observed by him at the time of solar eclipses, or at the time of the first visibility of the crescent.

Even up to the time of Pṛthūdaka, the accuracy in lunar astronomy was chiefly aimed at the time of eclipses. Thus in his commentary on the *Khaṇḍakhādyaka* IV, he makes the following introductory remarks :—

"All knowledge relating to (luni-solar) astronomy is desired by the wise (or cultured) specially for knowing the right instants of opposition or conjunction ; these instants are, however, not visible to the eye. Of other things such as *tithis*, *nakṣatras* and *Karaṇas*, as the planets, the Sun and the Moon, are not clearly observed, their beginnings and ends are not visible. Men see the agreement between calculation and observation at the times of solar and lunar eclipses. Hence the word of the astronomers is esteemed amongst men even in respect to such things as *tithis*, etc."[1]

Thus the chief aim of the ancient Indian astronomers was to calculate the eclipses accurately and the Moon was observed chiefly at lunar or solar eclipses, though the time for observation related also to the finding of the first visibility of the crescent. This latter phenomena did not perhaps lead them to directly observing the Moon's position at such times by using instruments.

Moon's Mean Motion

The practice of observing the Moon at the time of the eclipses alone led to the determination of the synodic month with the following results :—

(1) Mean synodic month according to the *Āryabhaṭīya*

$$= \frac{1577917500}{57753336-4320000} \text{ days,}$$

$$= 29.530582 \text{ days.}$$

1. बाहुल्येन पर्वकालार्थं सकलं ज्ञानमिष्यते शिष्टैः। तेषां च चक्षुषा दर्शनं नास्ति। अन्येषामपि तिथिनक्षत्रकरणानां तस्मात् तेषां शशिभास्करयोरव्यक्तेरंशत्वात्। शशिभास्कर-ग्रहणयोर्गणितदृष्टैक्यं लोकः पश्यति। तस्मात् तिथ्यादिष्वप्यर्थेषु दैवज्ञवाक्यं लोके माद्रियते।

(ii) The same according to the *Khaṇḍakhādyaka* =29 5305874 days

(iii) The same according to the *Brāhma-sphuṭa siddhānta* =29.530582 days

(iv) The same according to Ptolemy=29 da 31′ 50″ 8″ 20″ in sexagesimal units=29.5305927 days

The modern value according to Newcomb and Radau =29.5305881 days

Hence the *Khaṇḍakhādyaka* mean length appears to be the closest approximation.

The mean sidereal month must have been deduced from the mean synodic month and the year adopted Hence no comparison need be made of this element here

We will now consider the sidereal periods the nodes and the apogee of the Moon These are shown below -

According to	Sid. Per. of Moon s Apogee	Sid per of the Ascending Node
Āryabhaṭīya	3231.987079 da	6794 749511 da
Khaṇḍakhādyaka	3231 987844 da	6794 750834 da
Brāhma sphuṭa - Siddhānta	3232 73411 da	6792 25396 da
Ptolemy	3232.617656 da	6796 45571 da
Modern values (Lockyer)	3232.37543 da	6793 39108 da

Here also the Indian values show a closer approximation to the true values. Brahmagupta s figures representing the nearest approach.

Other Constants

So far the Indian values of the constants have been more accurate than the Greek ones, but as to the inclination of the Moon s orbit the Greek value is more accurate than the Indian value

Inclination of the lunar orbit

Indian value=4°30

Greek value=5°0'

Modern mean value=5°8 43' 427 (Brown)

This discrepancy confirms the conclusion, that the observation of the Moon was restricted to the time when she was near a node, either at solar or lunar eclipses, where a small error of observation magnified itself into about half a degree

We now turn to the parallaxes of the Sun and the Moon -

	Sun's Mean Hor Parallax	*Moon's mean Hor Parallax*
Āryabhaṭīya	3'55" 62	52'30"
Kaṇḍakhādyaka	3'56"5	52'42".3
Ptolemy	2'51"	53'34"
Modern values	0'8" 806	57'2" 79

As to the Sun s horizontal parallax, the ancients were of course totally wrong but in respect to that of the Moon their values were fairly approximate

We next consider the angular semi-diameters of the Sun and the Moon These are -

	Moon's Mean Semi diameter	*Sun's Mean Semidiameter*
Āryabhaṭīya	15'45"	16'29" 4
Khaṇḍakhādyaka (Brāhma-sphuṭa siddhānta)	16'0" 22	16'15"
Ptolemy	17'40"	15'40"
Modern values	15'33" 60	16'1" 8

Here also the Indian values are more accurate than the Greek values

Moon's Equations. The First Equation.

It remains now to consider the Moon s equations in ancient Indian astronomy As has been pointed out before, obser-

vation was up to the time of Brahmagupta, restricted to the time of eclipses perhaps also of syzygies

The modern form of the Moon's equations is

$$=377' \sin(nt-\alpha)+13' \sin 2(nt-\alpha)+\ldots\ldots\ldots$$

$$+76' \sin[2(nt-\theta)-(nt-\alpha)]+40' \sin 2(nt-\theta). \ldots\ldots\ldots$$ [1]

where nt=mean longitude of the Moon, α the longitude of the perigee, θ=longitude of the Sun.

Here the first two terms, viz., $377' \sin(nt-\alpha)+13' \sin 2(nt-\alpha)$, are due to elliptic motion about the Earth in one focus, the term $76' \sin[2(nt-\theta)-(nt-\alpha)]$ is known as the evection. We combine a part of the first term with the evection term and the expression for the equation of centre becomes $=301' \sin(nt-\alpha)+13' \sin 2(nt-\alpha)+\ldots\ldots\ldots+152 \sin(nt-\theta)\cos(\theta-\alpha)+40' \sin 2(nt-\theta)$

Now at syzygies and eclipses $\sin(nt-\theta)$ and $\sin 2(nt-\theta)$ will very nearly vanish. Hence according to modern astronomy at the syzygies and eclipses, the chief term of the Moon's equation$=301' \sin(nt-\alpha)$.

This according to the *Āryabhaṭīya*
$=300'\ 15'' \sin(nt-\alpha)$

" " *Khaṇḍakhādyaka*
$=296' \sin(nt-\alpha)$

" " *Uttara Khaṇḍakhādyaka*
$=301'.7 \sin(nt-\alpha)$

" " *Brāhmasphuṭasiddhānta*
$=293'\ 31'' \sin(nt-\alpha)$

" " Greek astronomy
$=300'\ 15'' \sin(nt-\alpha)$
very nearly.

Hence both the Greek and the ancient Indian astronomers were very near the true value of the Moon's equation at the syzygies and eclipses. Godfray in his *Lunar Theory*, page 107, observes "the hypothesis of an excentric whose apse has a progressive motion as conceived by Hipparchus served to calculate with considerable accuracy the circumstances of eclipses, and observations of eclipses requiring no instruments were then the only ones which could be made with sufficient exactness to test

1 The accurate values of the coefficients appear to be 377′ 1[illegible], 12′ 57″.11, 76′ 26″ and 39′ 30″ [illegible]

Greek value=5°0

Modern mean value=5°8 43 427 (Brown)

This discrepancy confirms the conclusion that the observation of the Moon was restricted to the time when she was near a node either at solar or lunar eclipses, where a small error of observation magnified itself into about half a degree

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Ptolemy	2 51"	53 34"
Modern values	0 8" 806	57 2" 79

As to the Sun s horizontal parallax the ancients were of course totally wrong but in respect to that of the Moon their values were fairly approximate

We next consider the angular semi-diameters of the Sun and the Moon These are

	Moon s Mean Semi diameter	*Sun s Mean Semidiameter*
Āryabhaṭiya	15 45"	16 29" 4
Khandakhādyaka (Brāhma sphuṭa siddhānta)	16 0" 22	16 15"
Ptolemy	17 40"	15 40"
Modern values	15 33" 60	16 1".8

Here also the Indian values are more accurate than the Greek values

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It remains now to consider the Moon s equations in ancient Indian astronomy As has been pointed out before obser-

vation was up to the time of Brahmagupta, restricted to the time of eclipses perhaps also of syzygies

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$=377' \sin (nt-\alpha)+13' \sin 2(nt-\alpha)+\ldots\ldots.$

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where nt = mean longitude of the Moon, α the longitude of the perigee, θ = longitude of the Sun.

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Hence both the Greek and the ancient Indian astronomers were very near the true value of the Moon's equation at the syzygies and eclipses. Godfray in his *Lunar Theory*, page 107, observes, 'the hypothesis of an excentric whose apse has a progressive motion as conceived by Hipparchus served to calculate with considerable accuracy the circumstances of eclipses, and observations of eclipses, requiring no instruments, were then the only ones which could be made with sufficient exactness to test

1 The accurate values of the coefficients appear to be 377',19".06, 12' 57".11, 76' 26" and 39' 30".

the truth or fallacy of the supposition." We next consider the second inequality of the Moon.

Moon's Second Inequality or Equation

In ancient times it was Ptolemy who first really found a second inequality of the Moon. According to Godfray (*Lunar Theory*, p. 107) "by dint of careful comparison of observations he (Ptolemy) found that the value of this second inequality in quadrature was always proportional to that of the first in the same place, and was additive or subtractive according as the first was so; and thus, when the first inequality was at its maximum or 5°1', the second increased it to 7° 40' which was the case when the apse line happened to be in syzygy at the same time."

It is well known that though Ptolemy discovered the second inequality in the Moon's motion he was not able to ascertain its true nature. His corrections in this case are true when at the quadrature the Moon's apse line passes through the Sun or it is at right angles to the line joining the Earth and the Sun.[1] In the general case his construction does not lead to the elegant form of the evection term as we know it now, nor does it lead to the nice form in which it was given by later Indian astronomers from the time of Mañjula (or Muñjāla, 854 Śaka era=932 A.D.).

As has already been pointed out, the early Indian astronomers from Āryabhaṭa to Brahmagupta aimed at accuracy in lunar calculation only for the eclipses and syzygies, and did not interest themselves about the Moon's longitude at the quadratures. Hence this second inequality is absent in the works of these makers of Indian astronomy, as also in the Pre-Ptolemaic Greek astronomy. This points to the conclusion that in both the earlier Indian and Greek systems of astronomy. the modes of observation of the Moon were copied from an earlier system of astronomy whether Babylonian or Chaldean. Even in the *Romaka Siddhānta* of the *Pañcasiddhāntikā*, there is no mention of evection.[2] Thus inspite of the transmission of a vague system of Greek astronomy, Indian astronomy as developed by Āryabhaṭa and Brahmagupta must be regarded as independent and

1. *Godfray's Lunar Theory*, pp. 108-110.

2. Vide the Summary in P.C. Sengupta's paper, "Āryabhaṭa the Father of Indian Epicyclic Astronomy." *Journal of the Department of letters*, vol. XVIII, *Calcutta University Press*.

original not only from this but also *from other considerations* It sought to correct the constants as were obtained from the Babylonian and the Greek systems as has in some cases been shown already.

Mañjula's Second Equation of the Moon (932 A.D)

We now take up in detail *Mañjula's second equation of* Moon It is however, necessary to say something about his first inequality

This is given in the form

$$\frac{-488 \sin (nt-\alpha')}{96+\frac{488}{120}\cos (nt-\alpha')} \text{ degrees}$$

where nt stands for the Moon's mean longitude α'—that of the apogee

Hence when $nt-\alpha'=90°$ the equation $= \frac{488°}{96} = 5°4 = 304$ showing an excess of 4 over the modern value

It is further necessary to modify the expression for the Moon's modern form of the equation by changing α to $180°+\alpha$ as in ancient Indian astronomy anomaly is measured not from the perigee but from the apogee

As to the positive or negative character of the "sine" and the "cosine" he gives the rule:—

> The mean planet diminished by its *ucca*, the apogee, aphelion or the Śighra, is called *Kendra* or mean anomaly; its "sine" from above six signs (180°) arises from half circles and are respectively positive or negative, and its "cosine" in different quadrants are respectively positive, negative, negative, and positive.[1]

The convention followed is that the "sine" is negative from 0° to 180° and positive from 180° to 360° of the arc and that the cosine is positive betwen 0° and 90°, negative between 90° and 270° and positive between 270° and 360°.

We may now symbolically express Mañjula's second inequality thus :-

$-(13^\circ\ 11'\ 35''-11^\circ)\times 8^p\ 8'\cos(\theta-\alpha)\times 8^p\ 8'\sin(D-\theta)$ where D stands for the Moon as corrected by the 1st equation; we leave out the correction to the Moon's daily motion as given in the stanzas quoted above.

The moon's new equation comes out to be

$$=-143'58'\cos(\theta-\alpha)\sin(D-\theta).$$

This, it will be seen, is exactly the modern form of the evection as combined with a part of the equation of apsis shown before. The difference in the main is that Mañjula's constant is 144', a quantity less by 8'. *In form the equation is most perfect, it is far superior to Ptolemy's, it is above all praise.* It is from this inequality, we trust, that Mañjula should have an abiding place in the history of astronomy. The next writer who gives the second equation is Śripati (1,028 A.D.).

Śripati's Second Inequality of the Moon

The following stanzas from Śripati's *Siddhānta Śekhara*, it is said, were communicated to Sengupta by Pandit Babua Misra. Though they are probably not very correct still the general meaning is clear. They carry the following sense :

> "From the Moon's apogee subtract 90°, diminish the Sun by the remainder left; take the "sine" of the

1. ग्रहः स्वोच्चोनितः केन्द्रं षड्भांदर्ध्वतो भुजः ।
धनर्णः पदशः कोटिर्धनर्णे धनात्मिका ॥

result, multiply it by 160′ and divide by the radius the result is called *caraphala* Put it down in another place, multiply it by *śara*(i e. R vers $(D-a)$ or versed sine of the Moon's distance from the apogee) and divide by the difference between the Moon's distance (hypotenuse) and the radius the result is called *parama (cara) phala* which is to be considered positive or negative according as the hypotenuse put down in another place is less or greater than the radius Multiply the sine of the Moon which has been diminished by the apparent Sun by the apparent *paramaphala* and divide by the radius the final result is to be called *caraphala* to be applied to Moon negatively or positively as the Moon minus the Sun and the Sun minus the Moon's apogee (diminished by 90°) be of opposite signs if these latter quantities be of the same signs the new equation should be applied in the inverse order by those who want to make the calculation of the apparent Moon agree with observation [1]

Symbolically —

$$\frac{160' \; R \sin[\theta-(a-90^\circ)]}{R} = caraphala$$

$$\mp \frac{160' \; R \sin[\theta-(a-90^\circ)]}{R} \times \frac{R \text{ vers } (D-a)}{H-R}$$

$= paramaphala$ according as H > or < R

The new equation

$$= \mp \frac{R \sin (D-\theta)}{R} \times paramaphala$$

---

1 त्रिभविरहितचन्द्रोच्चोनभास्वद्भुजज्या ।
गगननृपविनिघ्नी भक्तदज्या विभक्ता ॥
भवति चरफलाख्यं तत् पृथक् स्थं शरघ्नं ।
द्युमणिपतिकर्णत्रिज्ययोरन्तरेण ॥
परमफलमवाप्तं तद्धनर्णं प्रकल्प्ये ।
तुहिनकिरणकर्णे त्रिज्यकोनाधिकेऽथ ।
स्फुटदिनकरहीनादिन्दुतो या भुजज्या ।
स्फुटपरमफलघ्नी भाजिता त्रिज्ययाप्तम् ॥
शशिनि चरफलाख्यं स्पष्टहीनेन्दुगोलात् ।
तदृणमुत धनं चेत्तुल्यहीनार्कगोलम् ॥
यदि भवति हि साम्यं व्यस्तमेतद् विधेयम् ।
स्फुटगणितदृगैक्यं कर्तुमिच्छद्भिरेवं ॥

$$= \mp \frac{160' \, R \sin[\theta-(\alpha-90°)] \, R \text{ vers } (D-\alpha) \times R \sin (D-\theta)}{R \, (H-R) \times R}$$

$$= \mp \frac{160' \, R \cos(\theta-\alpha) \times R \sin(D \;\; \theta)}{R \times R} \times \frac{R \text{ vers } (D-\alpha)^1}{H-R}$$

This equation is a slightly modified one but practically the same in form as that of Mañjula, except that the constant here is 160′, greater than his by 16′ The constant is 160 also in Candraśekhara's form as we shall see later on We next consider the Moon s inequalities as given by Bhāskara II in his *Bijopanaya** a special work on these inequalities composed in the Saka year 1074 (=1152A D) two years after he had composed the *Siddhānta Śiromaṇi*

Bhaskara II on Moon's Inequalities

His preliminary statement runs thus —

> 112′ positive or negative representing the maximum difference have been found by me in the daily observed Moon (as calculated and as observed) at that point of the ecliptic where the arc from the *kadamba* (i e its pole) passing through the zenith cuts it.[1]

Thus for observing the Moon he selected the nonagesimal as the suitable point where the uncertainty about her parallax is zero and found ∓112′ of arc to be the maximum difference between her calculated and observed places

Mallabhatta perhaps a contemporary of Bhāskara II ascribed this difference to a supposed *Sighrocca* of the Moon Bhāskara in stanzas 9 13, refutes the existence of the Śighra in the case of the Moon the substance of his argument begins (i) that it is against the teaching of the *Surya siddhānta* and other accepted authorities (ii) that there is no variation of the apparent angular diameter of the Moon corresponding to this alleged *Śighra* and (iii) that planets having a *Śighra* have retrograde motion which is never the case with the Moon.

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*There is some uncertainty about this new fraction introduced by Śripati

1 लिप्ता विधोरर्कं महीमिता मे दृग्गोचरा प्रत्यहमं चितस्य ।
यदम्ब गोलगत सूत्रपाते क्रान्ती धनर्णवज्जुषो समस्यात् ॥

—*Bijopanaya* stanza 8

The reasons for his new equations are stated as follows—

When the Moon is situated at a quadrant ahead of the apogee and with the Sun at half a quadrant ahead of her the maximum discrepancy (of 112) is seen in the negative character

When the Moon is situated at three quadrants ahead of the apogee and with the Sun at half a quadrant behind her the maximum discrepancy (of 112) is seen in the positive character

When the eclipses of the Sun and the Moon take place at the apogee or the perigee of the Moon the Moon as corrected by the equation of apsis is seen to be without any new correction called *bija*

When the eclipses of the Sun and the Moon take place at the ends of the odd quadrants of the Moon s anomaly (measured from the apogee) the discrepancy is seen to be less by 34

When the Moon is at the apogee whether the Sun be ahead or behind her by half a quadrant the discrepancy amounts to be 34

The same discrepancy of 34 is observed when the Moon is at the perigee and the Sun is ahead or behind her by the same distance

Thus by analysis and synthesis and by repeated observations this variable correction has been devised by me let it be seriously considered by the learned [1]

Bhāskara here speaks of six cases and we consider them one after another —

The Moon s equations as modified to suit *siddhāntas* are given by

$$-301 \sin(nt-\alpha)+13 \sin 2(nt-\alpha)$$

$$-152 \sin(nt-\theta)\cos(\theta-\alpha)-40 \sin 2(nt-\theta)+$$

According to Bhāskara s *Siddhānta śiromani* the Moon s equation of apsis

$$= -\frac{31^{\circ}\ 36}{360}\times 3438 \sin(nt-\alpha)$$

$$= -301\ 46'\ 8 \sin(nt-\alpha)$$

this agrees well with the corresponding term of the modern equation. As Bhaskara takes in all the six cases $nt-\alpha=9ι°$, $270°$, $0°$ or $180°$, the second term of the equation of apsis vanishes

Case I

$$nt-\alpha=90°,\ nt-\theta=-45°,\ \theta-\alpha=135°,$$

Here the total equation of the Moon

$$=-301'-(76'+40')=-301'-116$$

This fairly agrees with Bhāskara s observation, the difference being only of 4′

Case II

$$nt-\alpha=270°,\ nt-\theta=45°,\ \theta-\alpha=225°,$$

the total equation of the Moon

$$=301'+76'+40'=301'+116'$$

This also agrees with Bhāskara s observation

Case III

$$nt-\alpha=0° \text{ or } 180°\ \ nt-\theta=0° \text{ or } 180°,\ \theta-\sigma=0° \text{ or } 180°,$$

the total equation$=0'$, this also agrees with Bhāskara s observation

Case IV.

$$nt-\alpha=90° \text{ or } 270°,\ nt-\theta=0° \text{ or } 180°\ \ \theta-\alpha=90° \text{ or } 270°,$$

---

1 तुङ्गादाद्यपदान्तस्थाद् विधोरर्के पदार्धगे ।
परमं चन्द्रवैषम्यं ऋणत्वेन समीक्ष्यते ॥ 20 ॥
तत् तृतीय पदान्तस्थात् पृष्ठगेऽर्के पदार्धगे ।
परमं चन्द्रवैषम्यं धनत्वेन समीक्ष्यते ॥ 21 ॥
चन्द्रतुङ्गे च नीचे च शशाङ्कार्कग्रहौ यदि ।
मन्दस्फुटगतश्चन्द्रो निर्बीजतुल्यमीक्ष्यते ॥ 22 ॥
ओजान्त्ययोर्विधोस्तुङ्गाच्चन्द्रशाङ्कार्कौ ग्रहौ यदि ।
चतुस्त्रिंशत् कलाहीनं वैषम्यं तु समीक्ष्यते ॥ 23 ॥
अग्रतः पृष्ठतो वाऽपि रवेश्चन्द्रे पदार्धगे ।
तुङ्गतुल्ये चतुस्त्रिंशत् कलावैषम्यमीक्ष्यते ॥ 24 ॥
एवं तन्नीचतुल्येऽपि वैषम्यं तावदेव हि ।
एवं व्यासात् समासाच्च प्रोक्तं पुनर्यत्नेन वेधनात् ॥
सर्वबीजमिदं क्लृप्तं मया सद्भिः समीक्ष्यताम् ॥ 25 ॥

the total equation $= \mp 301'$. This does not agree with Bhāskara's statement that the total equation

$$= \mp(301' \pm 78').$$

Case V.

$$nt-\alpha=0,\ nt-\theta=\pm 45^\circ,\ \theta-\alpha=\pm 45^\circ,$$

the total equation

$$=0'-76'+40'=-36' \text{ or } 0'+76'-40'=+36'.$$

This fairly agrees with Bhāskara's observation.

Case VI

$$nt-\alpha=180^\circ,\ nt-\theta=\pm 45^\circ,\ \theta-\alpha=180^\circ \mp 45^\circ$$

the total equation

$$=0'+76'+40'=0'+116', \text{ or } 0-76+40'=0'-36'.$$

This does not agree with Bhāskara's statement.

Bhāskara then states his first system of 24 equations corresponding to 24 sines in a quadrant to be 6', 13', 21', 27', 33', 39', 45', 51', 56', 61', 65', 68', 70', 72', 74', 75', 75', 76', 76' 77', 77', 78', 78', 78'.[1]

These equations, he says—"are negatively added to the equation of apsis when that is negative and positively added to the same when that is positive"[2]. In other words his new equations are complements of the equation of apsis, the two together being represented by

$$-301'\,46''.8 \sin(nt-\alpha) - 78' \sin(nt-\alpha)$$

$$\text{i.e., by } -379'\,46''.8 \sin(nt-\alpha).$$

Hence next states his second set of equations depending on $\theta - D$, to be 6', 9', 13', 17', 22', 24', 27', 30', 32', 33', 34', 34', 34', 33', 31', 29', 26', 24', 20', 16', 11', 8', 3', 0'[3] and says :

"These minutes are negative in the odd quadrants of the argument and are positive in other quadrants.[4]"

When the value of the argument is 15°, the equation is 17'.

"	"	"	45°	" 34'.
"	"	"	90°	" 0'.

1. *Bijopanaya*, 26-28.

2 ...फलं ऋणे ऋणं ।
धने धनं मन्दफलेन युग्मम् ॥ 28 ॥

3 *Bijopanaya* 29-32.

4 एताः कला ओजपदे ऋणं स्युर्धनं तदन्यत्र भवन्ति भूयः ।

Hence the new equation $= -34' \sin 2(\theta - D)$.
$= 34' \sin 2(D - \theta)$.

Here the symbol D stands for the Moon as corrected by the ancient Indian equation of apsis and its complement as given by Bhāskara. It is readily seen that Bhāskara is the first of all the Indian astronomers to detect the equation known as "Variation" His constant, 34', is less than the modern value by about 6', and cannot be considered as a serious error.

We now see that the sum-total of the Moon's equation as given dy Bhāskara

$= -379' \, 46''.8 \sin(nt - \alpha) + 34' \sin 2(D - \theta)$.

the evection term being totally absent. This is a serious defect, and Bhāskara's new equations would make the Moon generally more incorrect at the syzygies and eclipses than what the ancient Indian equation of apsis would do.

Perhaps late in life when he was 69 years old in 1105 of Śaka era (=1183 A.D.) he discovered the inapplicability of his new equations at the times of eclipses and in his *Karaṇa-kutūhala* he altogether omitted these new equations which he had given in his *Bījopanaya*.

As to Bhāskara's second inequality which is really the complement of the equation of apsis without the evection term, it is far inferior to that of Mañjula and of Śrīpati: as we have seen their form of the second inequality combines the complement of the equation of apsis and evection in the mathematically correct form. For the discovery of such a form of the equation as of these authors, very patient, careful and frequent observation must have been coupled with very careful and nice comparison of observed facts.

As to "variation" it was first discovered by Abul-Wefa in 976 A.D.[1] which was quite forgotten when Tycho-Brahe re-discovered it in 1580 A.D. Hence Bhāskara, in 1152 A.D., re-discovered it in India four centuries before Tycho.

Candraśekhara of Orissa on the Moon's Inequalities

In connection with lunar inequalities it is nesessary here to record what were the equations discovered or verified by M.M. Candraśekhara Simha of Orissā in the later half of the last century. He was educated in the orthodox Sanskrit fashion

1. Godfray's *Lunar Theory*, p. 114.

and had no acquaintance with English education His work *Siddhānta darpaṇa* was edited by Prof Jogeschandra Ray, late of the Cuttack College, in 1899[1] Candraśekhara in his work gives four equations of the Moon which are –

(1) The equation of apsis

(2) The Tungāntra equation or the complement of the equation of apsis in combination with evection

(3) The fortnightly equation or variation

(4) The Digamsa equation or the annual equation (i e $\frac{1}{1?}$ of the Sun's equation)

(5) The first equation is of the form

$$= \frac{[31^\circ\ 30' - 30' \cos(nt-\alpha)]3438 \times \sin(nt-\alpha)}{360^\circ}$$

$$= -300\ \ 49''5 \sin(nt-\alpha) + 4\ \ 46''5 \sin(nt-\alpha)\ \cos(nt-\alpha)$$

$$= -300\ \ 49''5(\sin nt-\alpha) + 2'\ 23''25 \sin\ 2(nt-\alpha)$$

It is seen that Candraśekhara wanted to correct the equation of apsis to the second order of small quantities as in all the Indian authors from Brahmagupta but Candraśekhara s form is correct though his contant is wrong

(2) His second equation is of the form

$$\frac{160' \times 3438 \sin[\alpha-(\theta+90^\circ)]}{3438} \times \frac{3438 \sin(D-\theta)}{3438}$$

$$\times \frac{\text{Moon's appt daily motion}^2}{\text{Moon s mean motion}},$$

$$= -160' \cos(\theta-\alpha) \sin(D-\theta)$$

$$\times \frac{\text{Moon's appt. daily motion}}{\text{Moon's daily mean motion}}$$

Here the constant is the same as that of Śrīpati discussed before The symbol the Moon as corrected by the equation of apsis It is readily seen that the constant of the first term of the equation of apsis is increased by 80 and that the constant of evection is taken at 80 In both the cases the error is about + 4

(3) Candraśekhara's third equation or Variation

$$= \frac{3438 \sin 2(D\ \ \theta)}{90} = 38\ \ 12'' \sin 2(D-\theta)^3$$

1 *Siddhanta-darpana*, V 100-114
2 Ibid VI 7 9
3 *Siddhanta-darpana* VI 11 12

where D means the Moon as corrected by the 1st and the 2nd equations Here the constant is wrong by $-1' 18''$

(4) His fourth equation or the annual equation $= \pm \frac{1}{10}$ of the Sun s equation of apsis,[1]

$$= \pm \frac{1}{10} \times \frac{12 \times 3438}{360} \sin \text{ (Sun s distance from the apogee)}$$

$= \pm 11' 27'' 6 \sin$ (Sun s distance from the apogee)

The modern value of the constant is 11' 10" Tycho found it to be 4 30" Horrocks' (1639) co efficient was 11' 51"

As Candraśekhara was aware of Bhāskara's *Bijopanaya* as also of the work of Śripati, his merit here lies in the discovery of the annual equation and correction to the constant of variation

Thus we have seen that so far as the luni-solar astronomy is concerned Indian astronomy is independent of Greek astronomy in respect of astronomical constants that Indian astronomy is generally more accurate than Greek astronomy and that Indian astronomers were not mere calculators [2] There were observers who verified and corrected the old astronomical constants as they came down from Āryabhata and Brahmagupta who also found independently all the principal equations of the Moon

1 Siddhānta darpana VI. 13
2. G R Kaye *Hindu Astronomy* p 60

— o —

Reference

P C Sengupta — *The Khandakhādyaka* 1934
Karl Manitius s edition — *Syntaxis*
Godfray — *Lunar Theory*

CHAPTER VII

Greek and Hindu Methods in Spherical Astronomy

Here we shall reproduce from Sengupta's paper a comparative account of the Greek and ancient Indian methods in Spherical Astronomy and to bring out the independence of the Indian Astronomers on this sub ect The views on this sub ect would necessarily differ from those of many European scholars such as Colebrooke and Bentley (early 19th century) to Kaye (early 20th century) Kaye wrote as follows in the *Journal of Asiatic Society of Bengal* 1919 No 3

> The methods by which (the rules) were obtained are buried in obscurity Braunmuhl[1] has stated "that the Indians were the first to utilise the method of pro ection in the Analemma of Ptolemy" It is intended to present the Hindu methods as clearly as possible and to show that Braunmuhl has not done sufficient justice to the Indian astronomers

As to Kaye, we shall show that his remark quoted above is due to the fact that he had to rely mostly on the English translation of the Sūryasiddhānta of Burgess, and perhaps he had no access to the works of Bhāskara II (1150 A D), who was the first to explain the ancient Indian methods clearly

Greek and Hindu Methods in Spherical Astronomy

Of the Greek methods in Spherical Astronomy, the history begins with elementary principles only from Euclid (300 B C) Even in Theodosius' *Sphaerica*[2] (about 153 B C) "there is nothing that can be called trigonometrical" Heath again says

1 Heath *Greek Mathematics* Vol II, p 291 Braunmuhl, *Geschichte der Trigonometrie* pp 38-42.

2, Heath *Greek Mathematics*, Vol II, p 250.

"the early spheric did not deal with the geometry of the sphere as such, still less did it contain anything of the nature of the spherical trigonometry. (This deficiency was afterwards made good by Menelaus's *Sphaerica*).[1] Hence the Greek spherical trigonometry began with Menelaus (90 A.D). His theorem in geometry is well-known—"If the sides of a plane triangle be cut by a transversal into six segments, the continued product of any three alternate segments, is equal to the continued product of the remaining three." *From this proposition he deduced the so-called "regula sex quantitatum" or the theorem, if the sides of a* spherical triangle be cut by an arc of a great circle into six segments, the continued product of the chords of the doubles of any three alternate segments is equal to the continued product of the chords of doubles of the remaining three segments" In plane geometry if the sides BC, CA, AB of a triangle be cut by any transversal at L, M, N, respectively, then we have

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1.$$

In spherics the theorem is:

$$\frac{\text{Chord } 2\,BL}{\text{Chord } 2\,LC} \cdot \frac{\text{Chord } 2\,CM}{\text{Chord } 2\,MA} \cdot \frac{\text{Chord } 2\,AN}{\text{Chord } 2\,NB} = 1$$

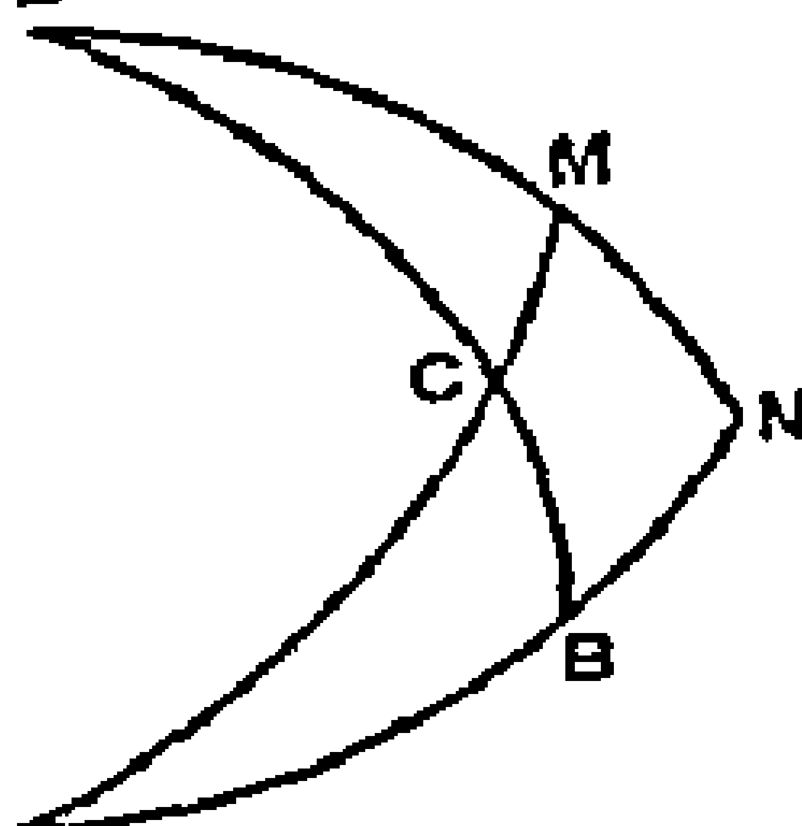


Fig. 5

Both these theorems are proved in Ptolemy's *Syntaxis* (Karl Manitius's edition, Vol I, pp. 45-51).

If R be the radius of the sphere on which the spherical triangle ABC is constructed, then the chord of the arc 2 BL= 2 R sin BL Hence Menelaus's theorem in spherics may be expressed as follows.

$$\frac{\text{Sin } BL}{\text{Sin } LC} \cdot \frac{\text{Sin } CM}{\text{Sin } MA} \cdot \frac{\text{Sin } AN}{\text{Sin } NB} = 1$$

This theorem is true for any spherical triangle

If $\angle B = AN = AM = 90°$ and L the pole of AB, then LMN is a secondary to the arc AB. There are four arcs of great *circles, taking any three as forming a spherical triangle and the* fourth as the transversal we readily get for the right-angled

1. A.A. Bjornbo, '*Studien über Menelaos' Sphärik*' in Abhandlungen Zur *Geschichte der Mathematischen Wissenschaften* for 1902, pp. 59 *et seq*, also Heath, *Greek Mathematics*, vol. II p. 261-73.

triangle ABC the relations —

(i) $\sin a = \sin b \sin A$

(ii) $\sin c = \tan a \cot A$

(iii) $\cos b = \cos a \cos c$

(iv) $\tan c = \tan b \cos A$

The above are some of the Napier's rules for a right angled spherical triangle deducible from Menelaus's theorem[1] They are generally sufficient in the case of such triangles In any spherical triangle however this theorem of Menelaus does not in any single step lead to any of the equivalents of the time altitude or altazimuth equations in spherical astronomy The ancient Indian methods though none of them are so highly finished as Menelaus's theorem yet are not less powerful in tackling the problems that arise in astronomy in connection with the apparent diurnal motion of the heavens The Greek or Ptolemaic method presents no further points of interest except in its application We now proceed to illustrate the ancient Indian methods and shall refer to the Ptolemaic method as occasion arises

Ancient Indian Methods
In Spherical Astronomy

In the Indian methods there is no general rule to follow It is by properties of similar right-angled triangles that a fairly complete set of accurate formulae are obtained These right angled plane triangles are classified under the names — Krānti kṣetras (triangles of declination) and Akṣa kṣetras (triangles of latitude) We consider the following problems —

Problem To find the time of rising on the equator of a length l of arc of the ecliptic measured from the first point of Aries

Let ω be the obliquity of the ecliptic and R. A the right ascension corresponding to the longitude l and δ the corresponding declination The Indian form of the equation is

1 Three more can be deduced similarly namely

(v) $\sin c = \sin b \sin C$

(vi) $\sin a = \tan c \cot C$

(vii) $\tan a = \cos C \tan b$

*$R \sin R A = \dfrac{R \sin l \times R \sin \omega}{R \cos \delta}$, where R is the radius of the sphere

Note — If R be the radius of the circle of reference the Indian trigonometrical functions for the arc θ, are (1) the 'sine,' (2) the 'cosine and (3) the versed sine They are respectively equal to $R \sin \theta$, $R \cos \theta$ and $R \text{ vers } \theta$

In the adjoining figure O is the centre of the armillary sphere, ϒQ ϒC are quadrants of the equator and the ecliptic respectively P is the celestial pole PCQ the summer solstitial colure Join Oϒ, OQ OP and OC

Let ϒS be $=l$, ϒM$=$R A, CQ$=\angle S\gamma M=\omega$ SM$=\delta$

Join OS, OM PSM is the secondary to the equator

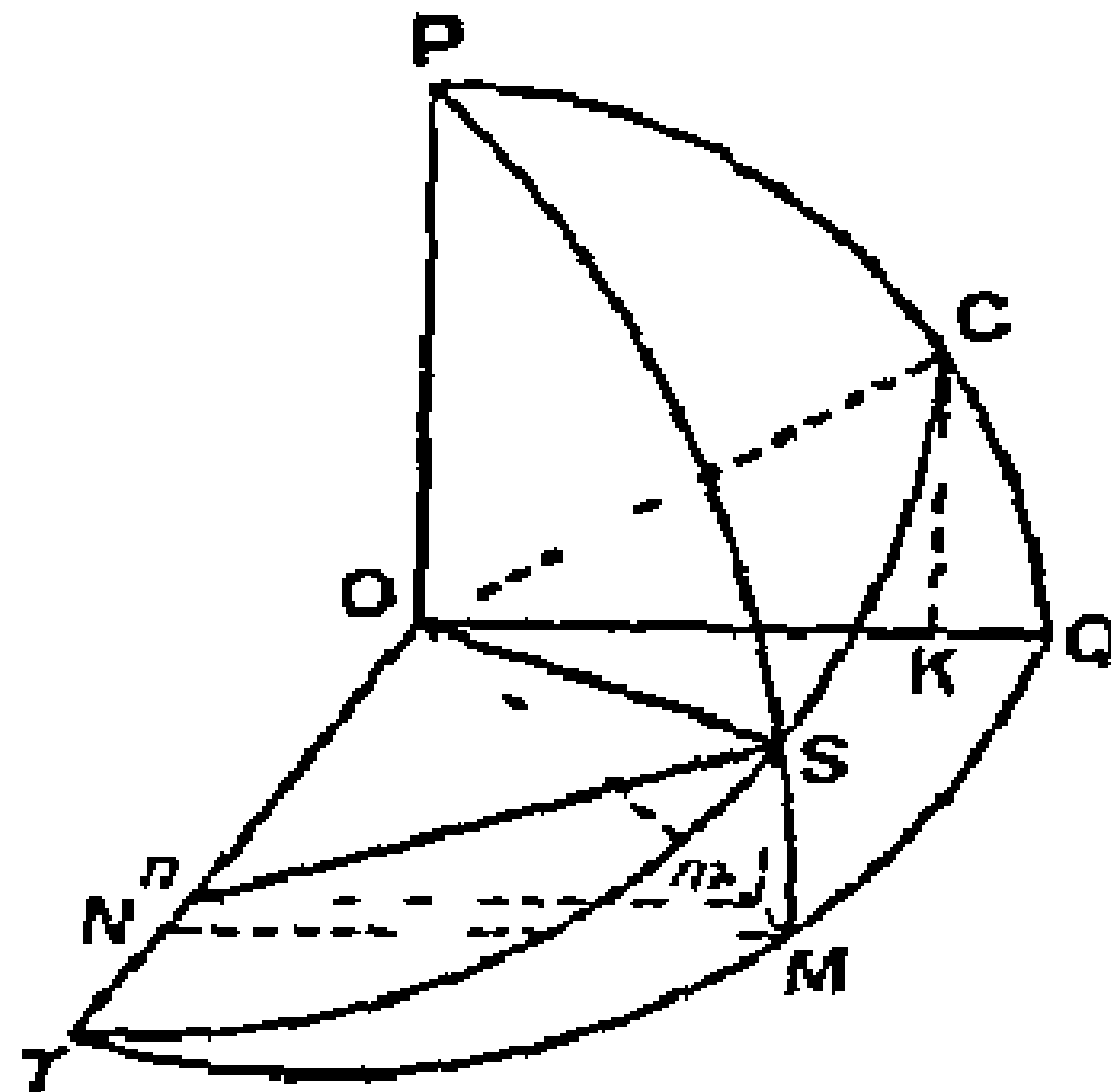


Fig 6

From C draw CK perpendicular to OQ From S draw *Sm* and *Sn* perpendicular to OM and Oϒ, respectively Join MN and from M draw MN perpendicular to Oϒ

Then the triangles *Smn* and CKO are similar They are called '*Krānti-kṣetras*'[1] or declination triangles,—similar right-angled triangles having one acute angle$=\omega$.

$$Sm : Sn = CK : OC$$

or $$R \sin \delta : R \sin l = R \sin \omega : R$$

$$R \sin \delta = \frac{R \sin l \times R \sin \omega}{R} \qquad \ldots (I)$$

The *Aryabhatiya Gola* 25 Varāhamihira in the *Pañcasiddhāntikā* (IV 92) states it in the form $2R \sqrt{\dfrac{(R^2 \sin^2 l) - R^2 \sin^2 \delta}{2R \cos \delta}}$

$= R \sin R A$ which is evident from the figure Brahmagupta's equation is identical with that of Āryabhaṭa (*BrSpSi* III 15 *Sūrya siddhanta* III 40-41

Also Bhāskara II *Grahagaṇita* cap VIII stanzas 54 55 is in agreement with Varāhamihira's forms

Greek Method

In the same figure[1] let PSC be the triangle and ♈MQ be the transversal Then Menelaus's theorem gives

$$\frac{\sin\ PM}{\sin\ MS}\times\frac{\sin\ S♈}{\sin\ ♈C}\times\frac{\sin\ CQ}{\sin\ QP}=1$$

or $$\frac{1}{\sin\delta}\times\frac{\sin l}{1}+\frac{\sin \omega}{1}=1$$

or $$\sin\ \delta=\sin\ l\times\sin\ \omega$$

Indian Method

Again by the Indian method from the same two similar triangles we get

$$mn\quad nS=OK\quad OC$$

or, $$mn\quad R\sin l=R\cos\omega\quad R$$

$$mn=\frac{R\sin l\times R\cos\omega}{R}$$

Again MN mn=OM Om

i e, R sin R A mn=R R cos δ

$$R\sin R\,A=\frac{R\sin l\times R\cos\omega}{R\cos\delta}$$

Greek Method

Take PQM for the triangle and ♈SC for the transversal

Then, $$\frac{\sin\ PC}{\sin\ CQ}\times\frac{\sin\ Q♈}{\sin ♈M}\times\frac{\sin MS}{\sin\ SP}=1$$

or $$\frac{\cos\omega}{\sin\omega}\times\frac{1}{\sin\ R.A}\times\frac{\sin\ \delta}{\cos\ \delta}=1$$

or $$\sin R\,A=\tan\delta\cot\omega$$

The Indian form of the equation is different from that of Ptolemy's. It is also better for the purpose of calculation

Note —From the same two similar triangles we have

$$On\quad ON=R\cos\delta\quad R$$

$$On\quad R\cos l=\frac{R\cos R\,A\times R\cos\delta}{R}\qquad (3)$$

Again $$\tan R.A.=\frac{mn}{On}$$

$$=\frac{R\sin l\times R\cos\omega}{R\times R\cos l}\qquad (4)$$

Again mn Sm=OK KC

1 Manual on [illegible] of Spherics I, 51 53.

or $\quad mn = \frac{R \sin\delta \times R \cos\omega}{R \sin \omega}$

$\therefore \quad R \sin R.\ A. = \frac{MN}{mn} \times mn = \frac{R}{R \cos\delta} \times \frac{R \sin\delta \times R \cos\omega}{R \sin \omega}$ (5)

Problem II :—

Sidereal Time-intervals

Indian Method

The problem discussed above provides the method of finding the sidereal time-intervals in which the signs of the zodiac rise on the equator. To find the corresponding times at any latitude ϕ, it is necessary to calculate and apply what is the ascensional difference due to the elevation of the celestial pole. This ascensional difference is called '*carakala*' or the variation in the length of half the day. The 'sine' of this '*carakala*' is called '*carajyā*.' If *ch* denotes this '*carakāla*.'

then.[1] $R. Sin\ ch = \frac{R \sin\phi \times R \sin \delta \times R}{R \cos \phi \times R \cos \delta}$

Just as in the solution of the previous problem. the declinational triangles or '*Krānti Kṣetras*' were constructed and used. so in the solution of this and other problems another set of similar triangles were conceived and constructed and were given the name '*Akṣa kṣetras*.'[2]

Let NPZH be the meridian (Fig 7). NOH the north-south line passing through the observer O P the celestial pole. OQ the trace of the equator on the meridian plane. Z the zenith. Join OZ. From Q draw QM perpendicular to OZ. Then the triangle QOM is an '*Akṣa-kṣetra*' or a latitudinal right-angled triangle. as ∠QOM = ϕ, the latitude of the station Another '*Akṣakṣetra*' is thus conceived. in the same figure. let P, P′ be the north and south celestial poles. N. the north point. AB A′B′ the diurnal

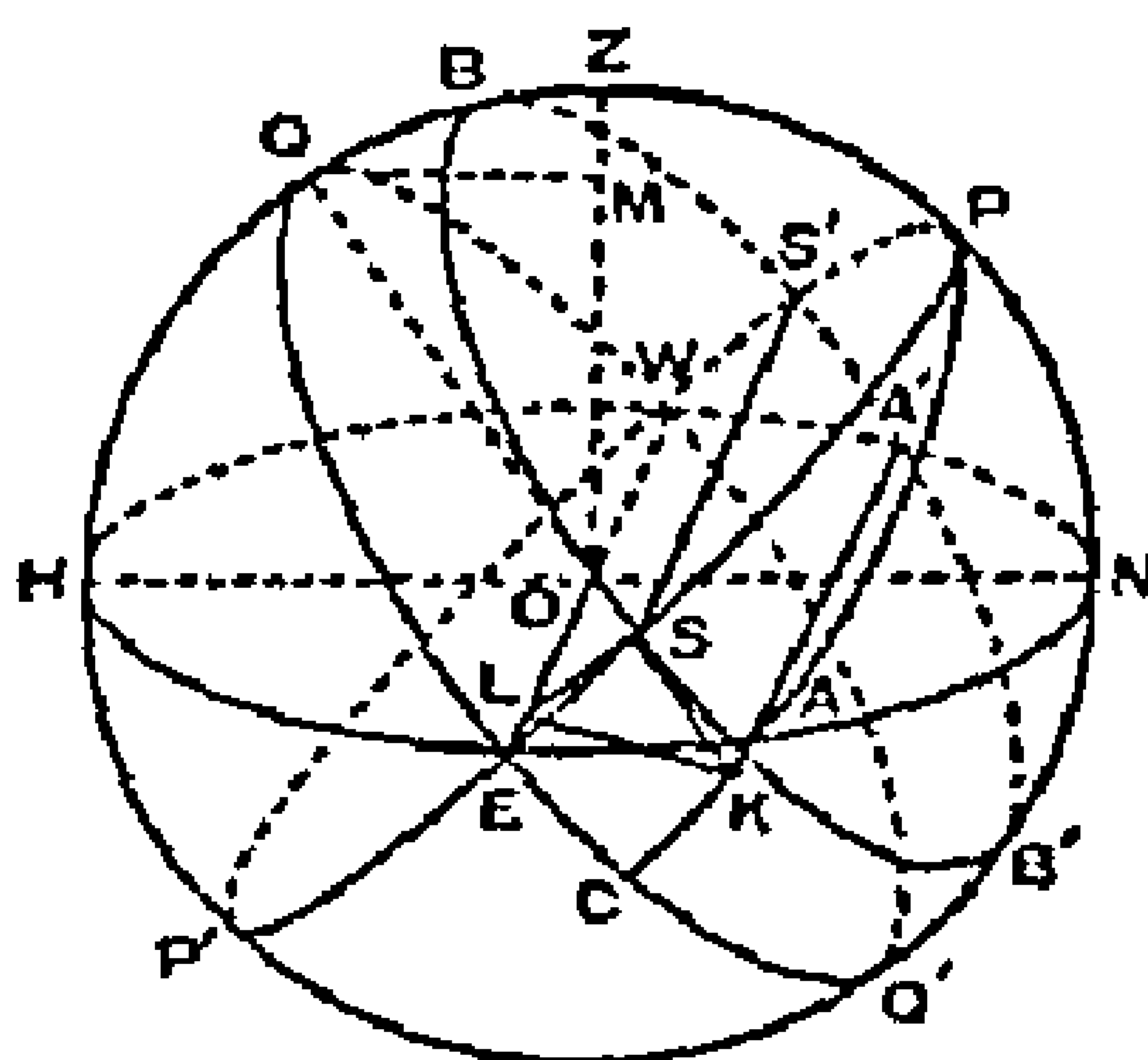


Fig 7

1. *Āryabhaṭīya*, Gola, 26. *Pañca-siddhāntikā*, IV, 34 *Brahmasphuṭasiddhānta*, II, 57-58, *Sūrya-siddhānta*, II, 91, Grahagaṇita, VIII, 45-47

2. Bhāskara, Golādhyāya (Wilkinson and Bapū Deva Śāstri's tr.) PP. 173 76- also, Bhāskara, Grahagaṇita, Ch. IX, 13-17.

circle of a heavenly body with declination δ NEHW the horizon PEP W the six O clock circle Here AA' the line of intersection of the diurnal circle with the horizon is called the '*udayasta sūtra* [1] (or the thread joining the rising and setting points) SS' the line of interesection of the diurnal circle and the six o' clock circle is the horizontal diameter of the diurnal circle From S draw SK and SL perpendiculars respectively to AA' and EW Join KL

Now since PN=ϕ the latitude of the station in the small right angled triangle KLS the $\angle$ KLS is also=ϕ

$$SK : SL = QM : MO$$

or $$SK=\frac{SL\times QM}{MO}=\frac{R\sin\delta\times R\sin\phi}{R\cos\phi}$$

Now SK[2] is a "sine in the small circle AB A'B' of which the radius is R cos δ this sine reduced to the equator (radius R) is the sine of *cara*

$$R\sin ch = R\sin EPA$$
$$=\frac{R\sin\delta\times R\sin\phi\times R}{R\cos\phi\times\cos\delta}$$

Greek Method

Let[3] the arc PA be produced to meet the equator at C Take PCQ for the triangle and EAN for the transversal Then we get

$$\frac{\sin PA}{\sin AC}\times\frac{\sin CE}{\sin EQ}\times\frac{\sin QN}{\sin NP}=1$$

or $$\frac{\cos\delta}{\sin\delta}\times\frac{\sin CE}{1}\times\frac{\cos\phi}{\sin\phi}=1$$

$$\sin CE=\sin ch=\frac{\sin\phi\times\sin\delta}{\cos\phi\times\cos\delta}$$

Note—The perpendicular distance between AA' and Ew is called the sine of the amplitude or the *Agra* which is thus calculated —

$$KL : LS = QO : OM$$

[4]R sin amplitude= *Agrā* =KL

$$=\frac{LS\times QO}{OM}=\frac{R\sin\delta\times R}{R\cos\phi}$$

It is now evident that the ancient Indian method is different

1 Bhāskara *Gola* VII 39

2 This is called by the name *kujyā* or *kṣitijyā* i e earth sine Aryabhaṭa *Gola* 26 Brahmagupta II, 57 *Sūrya-siddhānta II 61 etc*

3 Manitius, ib d, p 84

4 Aryabhaṭa *Gola* 30 etc.

from the Greek method in this case also. As the triangle KLS is difficult to show in the diagram, it is shown in its projection on the meridian plane in Burgess's translation of the "*Sūrya-siddhānta.*"(page 232)and also in Wilkinson and Bāpūdeva Śāstrī's translation of the '*Siddhānta Śiromaṇi.*' p. 175. This has led Braunmühl to assume that the Indian method of arriving at the equation of ascensional difference and some other equations of spherical astronomy has its origin in the Analemma of Ptolemy. *A careful study, however, does not justify the identification of* Indian methods with the graphic *method of the Analemma,* which is deduced from the projections of the position of a heavenly body on the meridian prime vertical and the horizon. It is being presently shown that what was done out of difficulty in drawing the figures properly has been taken by Braunmühl as a Greek connection.

Problem III[1] :—

To find the "Time-altitude" Equation

If from any point S on the diurnal circle a perpendicular be drawn to the *Udyasta-Sūtra* spoken of before, this perpendicular is called the *cheda* or '*iṣṭahṛti.*' The perpendicular from S on the horizon is called '*Śanku*'[2] the sine of the altitude. The line joining the foot of the '*Śanku*' and that of the perpendicular on the '*Udayāsta-Sūtra*' goes by the name of '*Śankutala*' and this *Śankutala* lies to the south of the '*Udayasta-Sūtra*' during the day.

In this figure (Fig 8) if AA' be the 'Udayāsta-Sūtra' or the intersection of the diurnal circle and the horizon, and S a point on the diurnal circle denoting a position of the Sun. SK SL perpendiculars on AA' and the horizon respectively; SL is called the '*Śanku,*' SK the '*cheda*' and LK, the '*Śankutala*'. In this triangle KSL, the angle KSL was recognised to be the latitude of the station.

Thus the triangle SKL is not taken in its projection on the meridian plane. The side SK is taken as formed of two parts.

1. Āryabhaṭa could not arrive at the true equation. Cf. Gola 28. The correct rules occur in *Pañcasiddhāntikā*, IV, 42, 44, *Brahmasphuṭasiddhānta*, III, 36-38, 26-40; *Sūryasiddhānta*, III, 34-35.

2. Bhāskara says : भस्थानाल्लम्बः शंकुः । तस्यैवमूलादुदयास्तसूत्रात् छेदो भवति ॥ "*Gola*, VIII-39-41, Āryabhaṭa uses the term शङ्क्वग्रम्" *Gola*, 29.

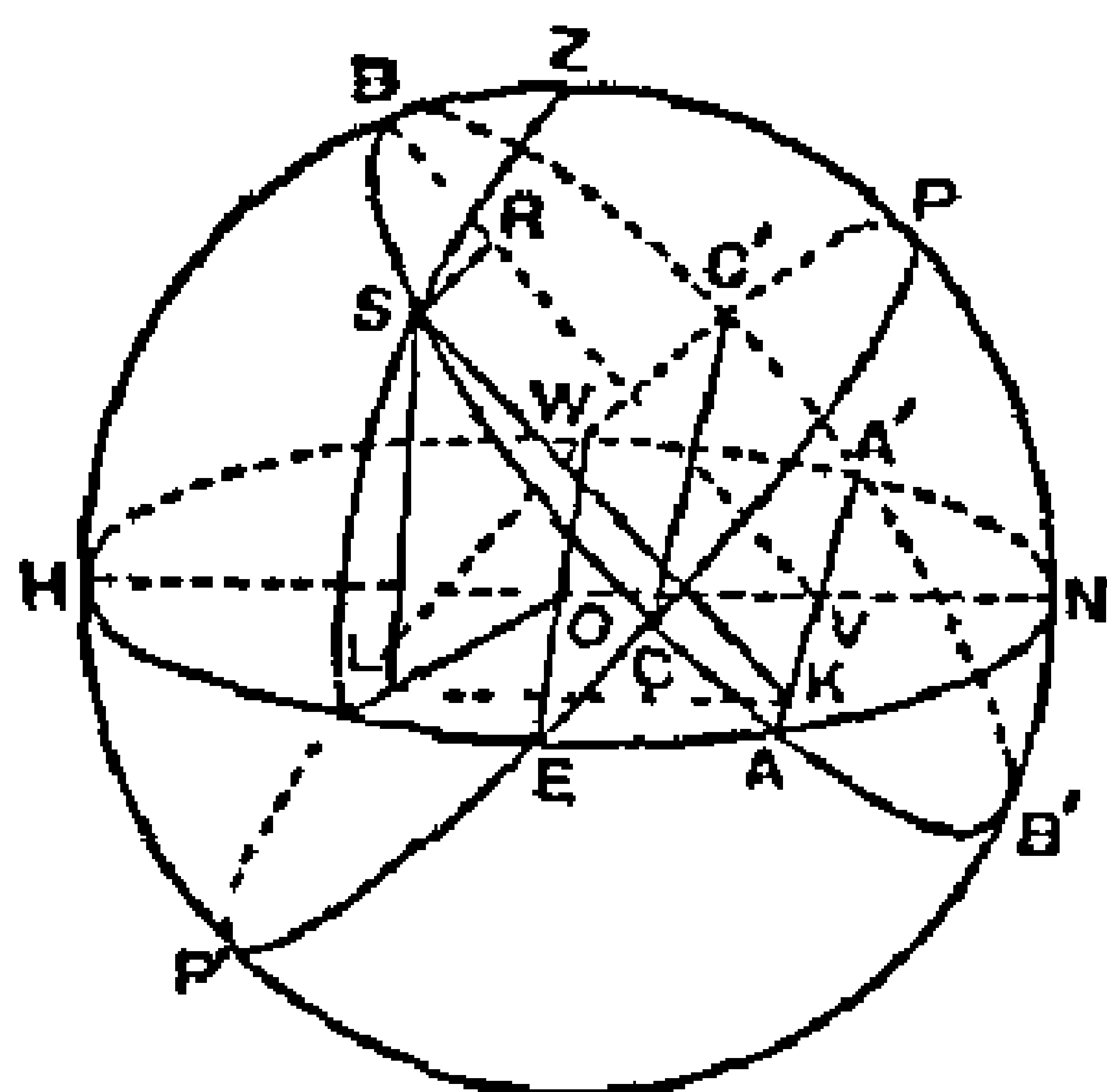


Fig 8

Let CC' be the line of intersection of the diurnal circle and the 'six o'clock 'circle EPW. Let SK cut CC' in M Then

$$SK = SM + MK$$

Here SM, the 'sine' in the diurnal circle of the complement of the hour angle is given a distinct name '*Kala*[1] and MK as explained before is known by the name '*Kujya*' This '*Kala* is constructed from the point S in the diurnal circle Thus the triangles like SKL were not taken in their projections on the meridian plane as Braunmühl would suggest.

From the triangle KSK, we get

'*Cheda*' : '*Śaṅku*' = R : R cos ϕ where ϕ is the latitude of the observer,

'*Śaṅku*' is here = R cos Z, Z being the Sun's zenith distance.

$$\therefore \quad \text{'cheda'} = \frac{R \cos Z \times R}{R \cos \phi}$$

Now '*Cheda*' = radius of the diurnal circle + *Kujya*—versed sine of the hour-angle in the diurnal circle O' B + O' V − BR,

$$= R \cos \delta + \frac{R \sin \delta \times R \sin \phi}{R \cos \phi} - \frac{R \text{ vers } H \times R \cos \delta}{R}$$

As in the previous problem *Kujya* = SK = $\dfrac{R \sin \delta \times R \sin \phi}{R \cos \phi}$

or
$$\frac{R \cos Z \times R}{R \cos \phi} = \frac{R \cos \delta}{R} \left\{ R + \frac{R \sin \delta \times R \sin \phi}{R \cos \phi} \times \frac{R}{R \cos \delta} - R \text{ vers } H \right\}$$

The above equation simplified becomes

$$\cos Z = \sin \delta \sin \phi + \cos \delta \cos \phi \cos H$$

In this connection we consider the altazimuth equation by the Indian method

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1 Bhaskara's *Grahagaṇita*, VIII, 53

O is the middle point of CC' or it is the centre of the diurnal circle ABB'

[1]*Problem IV* :—

Indian Method *The Altazimuth Equation*

Let a denote the azimuth of the Sun from the south. In the same triangle SKL in the same figure, we have,

LK : SL = R sin ϕ : R cos ϕ

or, '*Śankutala*' : '*Śanku*' = R sin ϕ : R cos ϕ

$\therefore$ '*Śankutala*' $= \dfrac{R \cos Z \times R \sin \phi}{R \cos \phi}$

Now '*Śankutala*' is made up of two parts, namely, '*Bāhu*' and '*Agrā*,' of which the former is the distance of L from the observer's East-West line; the '*Agrā*' has been already found.

Here '*Bāhu*' $= \dfrac{R \sin Z \times R \cos a}{R}$ and '*Agrā*' $= \dfrac{R \sin \delta \times R}{R \cos \phi}$

$\therefore$ '*Śankutala*' = '*Bāhu*' + '*Agrā*'

or $\dfrac{R \cos Z \times R \sin \phi}{R \cos \phi} = \dfrac{R \sin Z \times R \cos a}{R} + \dfrac{R \sin \delta \times R}{R \cos \phi}$

or $R \sin\delta = \dfrac{R \cos \phi}{R} \left(\dfrac{R \cos Z \times R \sin \phi}{R \cos \phi} - \dfrac{R \sin Z \times R \cos a}{\phi} \right)$

which is easily seen to be equivalent to

$\sin \delta = \cos Z \sin \phi - \sin Z \cos \phi . \cos a$

Greek Method

Ptolemy[2] has also a method of finding the Sun's altitude at any hour of the day. His method is as follows :—

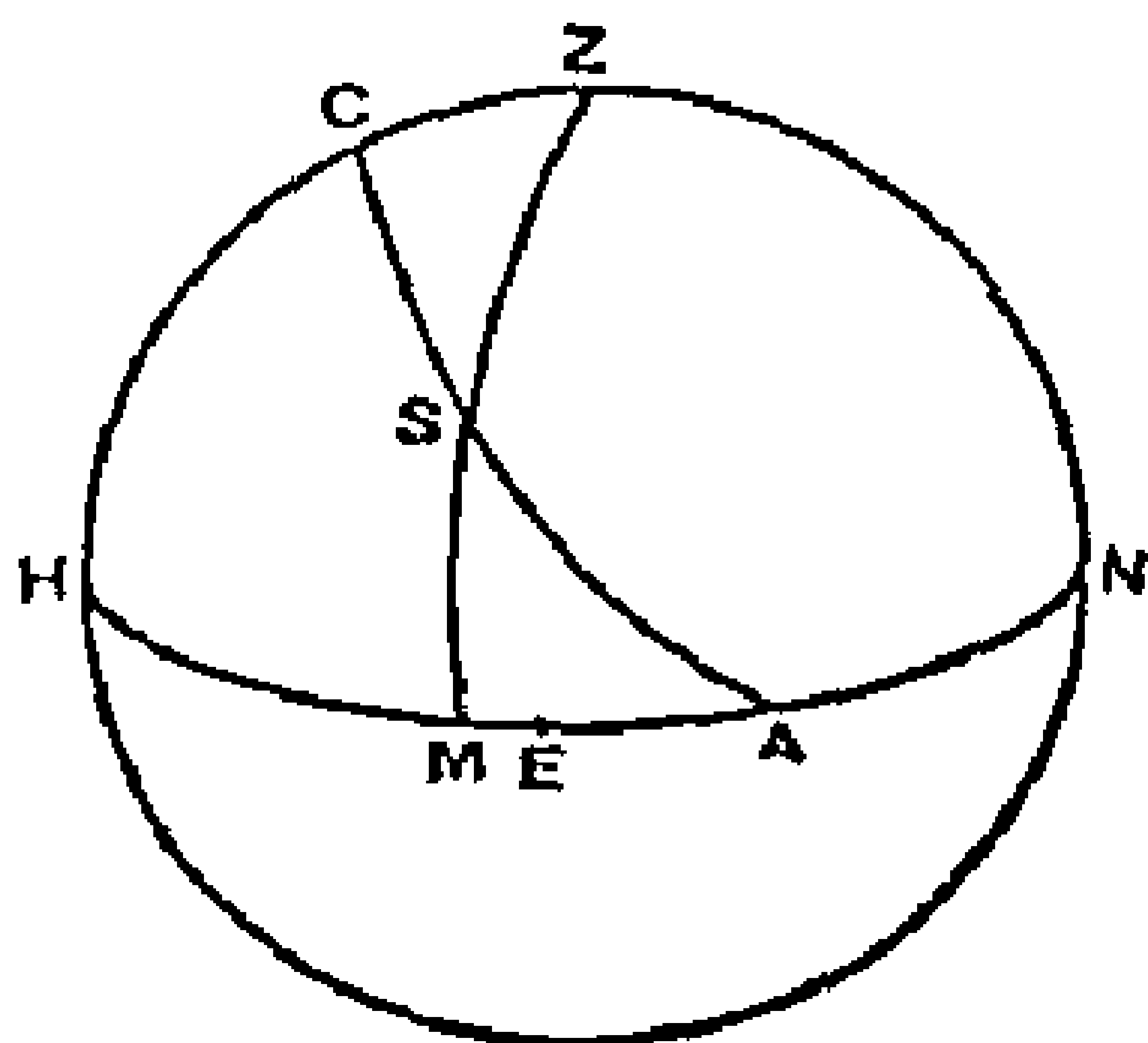


Fig. 9

(i) He would find by means of his tables for the times of risings of the signs of the zodiac, the orient ecliptic point. (ii) He would then find the culminating point of the ecliptic. (iii) He would finally apply Menelaus's theorem in spherics thus :—

Let ASC be any position of the ecliptic,(Fig. 9)NZC the

1. The equivalent of this in a particular case, is first found in *Brahma-sphuṭasiddhānta*, Ch. III, 54-56 Cf. *Sūryasiddhānta*, III, 28-31, also Bhāskara *Grahagaṇita* IX 50-52.

2. Manitius, ibid, pp 119, 19.

meridian, NAMH the horizon Z, the zenith and S the Sun Here the celestial longitudes of C, S and A are taken to be known, hence ZC and CH are also known

Now take ZCS for the triangle and HMA to be the transversal, we then have by Menelaus's theorem

$$\frac{\sin ZH}{\sin HC}\times\frac{\sin CA}{\sin AS}\times\frac{\sin SM}{\sin MZ}=1$$

$$\text{or } \sin SM=\frac{\cos CZ\times\sin AS}{\sin CA}$$

It is thus clear that Ptolemy had no direct method for connecting the Sun's altitude and the hour angle This method is workable for the problem "given time, find the altitude" but is not workable in the converse problem, besides the calculation of the longitudes of A and C is very cumbrous.

Again when EA has been found out, taking ZHM for the triangle and CSA for the transversal we get

$$\frac{\sin HA}{\sin Am}\times\frac{\sin MS}{\sin SZ}\times\frac{\sin ZC}{\sin CH}=1,$$ whence and thence HM, the azimuth can be found The method is here also cumbrous, there being no direct connection between altitude and azimuth, besides the time element is not avoided

The Analemma of Ptolemy
and the Indian Method.

When the Sun's declination is zero and his hour-angle, is H Zeuthen[1] following the method of the 'Analemma' of Ptolemy, as explained by Braunmuhl[2] has deduced the following equations

(1) $\cos Z=\cos H\ \cos\phi$

(2) $\tan \alpha=\dfrac{\tan H}{\sin\phi}$

To these two, Heath following Braunmuhl adds

(3) [3]$\tan ZQ=\dfrac{\tan H}{\cos\phi}$

1. Heath *Greek Mathematics* Vol II, pp 290-91
Zeuthen *Bibliotheca Mathematica* 1s 1900 pp 23-27

2. Braunmühl ibid, pp 12 13

3 The Indian form of this equation is $R\ \text{Sin}\ ZQ=\dfrac{R\ \text{Sin}\ H\times R}{\dfrac{\sqrt{R^2-R^2\cos^2 H\times R^2\ \text{Sin}^2\phi}}{R}}$

Bhaskara's *Golādhyāya*, Com. on VIII 67

where Z is the zenith and Q is the point of intersection of the prime vertical and its secondary passing through the Sun and the north-south points

Zeuthen[1] points out that later in the same treatise Ptolemy finds the arc 2β described above the horizon by a star of given declination δ' by a procedure equivalent to the formula

(4) $\cos \beta = \tan \delta' \tan \phi$

With regard to the 'Analemma' of Ptolemy it may be noted, as Heath[2] says that "the procedure amounts to a method of graphically constructing the arcs required as parts of an auxiliary circle in one plane" Many things may be, in practice done graphically far more easily than by the theoretical method Besides, no theoretical calculations occur in the 'Analemma' Zeuthen[2], following the method of this work, has deduced in the general case, the two equations

(5) $\cos Z = (\cos \delta \cos H + \sin \delta \tan \phi) \cos \phi$

(6) $\tan a = \dfrac{\cos \delta \sin H}{\dfrac{\sin \delta}{\cos \delta} + (\cos \delta \cos H + \sin \delta \tan \phi) \sin \phi}$

These equations are suggested to a modern reader from a study of the figures in the 'Analemma' But neither in this work nor in the '*Syntaxis*' are they to be found With regard to the first four formulae, it is possible that they were recognised by Ptolemy With regard to the last two Zeuthen[3] remarks "mais le texte nen contient rien" and they were certainly not recognised by Ptolemy

Besides the tangent function is wholly absent in Greek trigonometry They are also different in form from those arrived at by the Indian method as explained before Thus, it is clear that the Indian methods are in no way connected with the method of the 'Analemma'

Even taking for granted that the Indians followed a method of projection much allied to the method of the Analemma' there is no adequate reason for assuming that their method is derived from any Greek source Analogy and precedence do not necessarily constitute originality—there is still the chance of a remoter origin from which both the systems drew their inspiration The method of the Analemma as has been already stated presents a

1 2, Bjo nbo loc cit p 86.

3 Zeuthen loc cit p 27

the *secondary to the equator* cutting it at E'. Both the above astronomers were content with the idea that AE=AE', or that AE=the declination of the point A of the ecliptic which is 90° ahead of S in the above figure. This idea continued till the time of Bhāskara II (1150 A. D.) who found out the correct equation.

He recognised that CS, the declination of S=PP'; P'EH is then the horizon of the station whose north geographical latitude is CS. Also, the 'sine' of EA is the '*Agrā*' or the sine of the amplitude of the point A for the latitude CS.

$$\therefore R \sin EA = \frac{R \sin AE' \times R}{R \cos CS} = \frac{R \sin (90^\circ + \gamma S) \times R \sin \omega}{R} \times \frac{R}{R \cos CS}$$

or $$R \sin EA = \frac{R \sin (90^\circ + l) \times R \sin \omega}{R \cos \delta}$$

where l stands for γS and δ for CS.

Greek Method :

We give below the Ptolemy's method in a slightly modified form[1]. Let SHA be the triangle and γCE be the transversal; then we have,

$$\frac{\sin SC}{\sin CH} \times \frac{\sin HE}{\sin EA} \frac{\sin A\gamma}{\sin \gamma S} = 1$$

or $$\frac{\sin \delta}{\cos \delta} \times \frac{\sin 90^\circ}{\sin EA} \times \frac{\sin (90^\circ + l)}{\sin l} = 1$$

$$\therefore \sin EA = \frac{\sin \delta \times \sin (90^\circ + l)}{\cos \delta \times \sin l},$$

which is readily *transformed into Bhāskara's equation.* The originality of Bhāskara would be readily admitted.

Problem VI—

To find the Angle between the Ecliptic and the Horizon

Indian Method :

(A) Āryabhaṭa's method. It consists of the following[2] steps :—

(1) Determination of the orient point of ecliptic.

(2) Finding the sine of its amplitude.

1. Manitius, *ibid*, Book I, pp. 104-06.
2. Āryabhaṭa, *Gola*, 33 : *Sūryasiddhānta*, V. 5-6.

(3) Determination of the culminating point of the ecliptic from the hour-angle of the Sun

(4) Finding the declination of the culminating point of the ecliptic

Having obtained the above elements his rule can be followed thus

In this Fig 11 NZH is the meridian HMEAN the horizon CN'A the ecliptic If N be the nonagesimal or the highest point of the ecliptic the altitude of N is the inclination of the ecliptic to the horizon

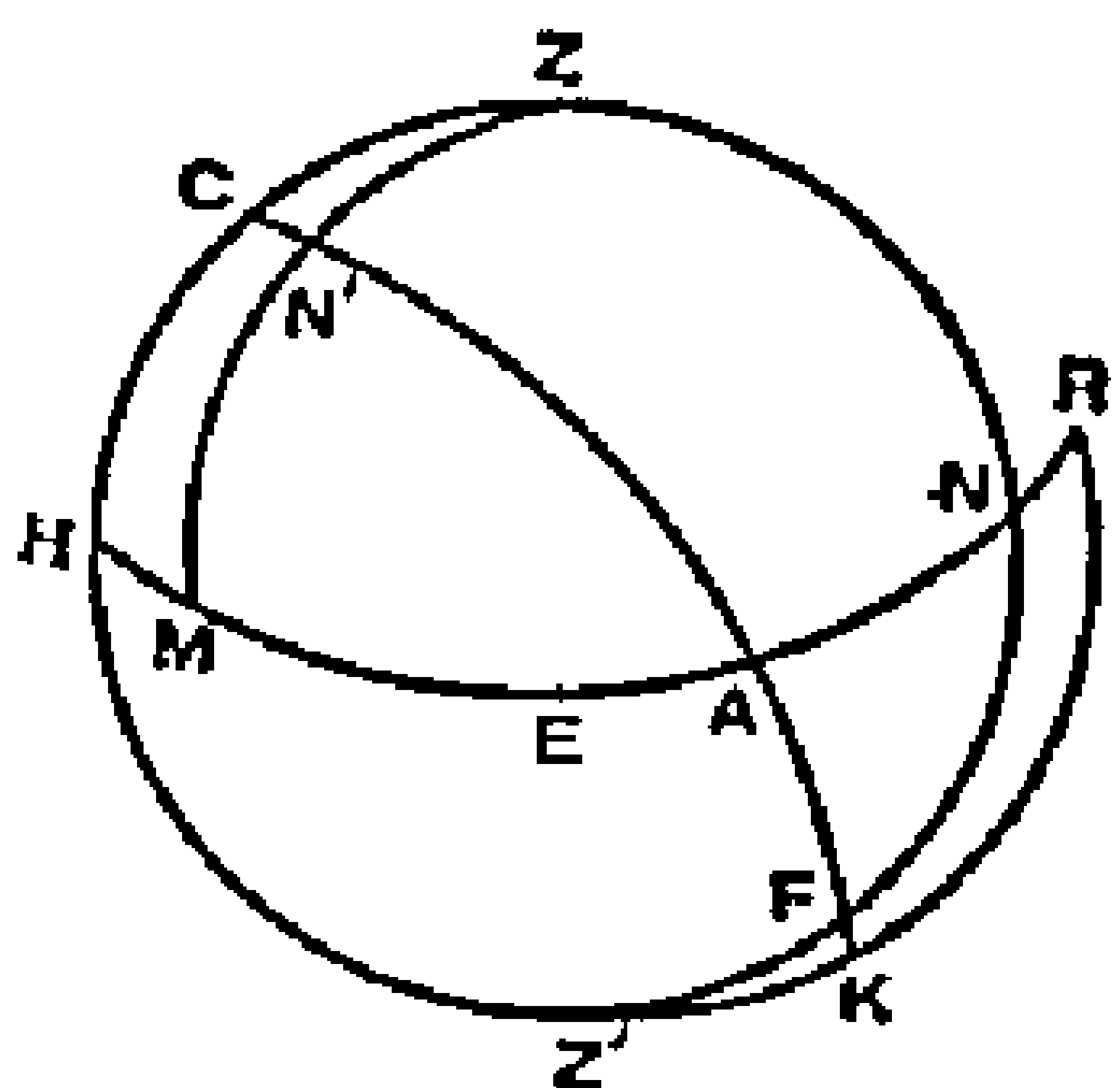


Fig 11

Let ZN M be the vertical through N, meeting the horizon at M

When the time is given the longitudes of A and C can be found out, from which CZ the zenith distance of C and EA the amplitude of the orient ecliptic point can be determined

Here HM=EA

According to Āryabhaṭa,

$$R \sin CN = \frac{R \sin CZ \times R \sin HM}{R}$$

$$\text{and } R \sin ZN = \sqrt{(R \sin CZ)^2 - (R \sin CN)^2}$$

This is only an approximate rule As expressed here

$$R \sin ZN = \frac{R \sin CZ \times R \cos HM}{R} \text{ approximately}$$

$$= \frac{{}^1 R \sin CZ \times R \cos HM \times R}{R \times R \cos CN'} \text{ accurately}$$

$$= \frac{R \sin CZ \times R \cos HM}{R \cos CN'}$$

(B) The method of Brahmagupta[2]

Brahmagupta would also first determine the orient ecliptic

1 This correction was perhaps first noticed by Ranganātha (1603 A D) in his commentary to the Sūryasiddhānta

2 Br Sp Si 4 3

point A. Then he subtracts 90° from the longitude of A. Thus having the longitude of N′, he next finds the part of the day elapsed of N′; from which by the time-altitude equation discussed above, he finds ZN′. This is of course more accurate than that of Āryabhaṭa. Bhāskara[1] here follows Brahmagupta.

Greek Method:

Let the ecliptic CN′A cut the lower half of the meridian at F. Ptolemy takes AK along the ecliptic=90° and AR along the horizon=90°; then the great circle passing through R and K passes through the nadir Z′. Now take Z′FK for the triangle and ANR for the transversal, then by Menelaus's theorem.[2]

$$\frac{\sin FN}{\sin NZ'} \times \frac{\sin Z'R}{\sin RK} \times \frac{\sin KA}{\sin AF} = 1$$

$$\therefore \sin RK = \frac{\sin FN}{\sin AF} = \frac{\cos FZ'}{\sin AC} = \frac{\cos CZ}{\sin AC} = \frac{\sin CH}{\sin AC}$$

$$\text{or } \sin MN' = \frac{\sin CH}{\sin AC}.$$

Here Ptolemy's equation is simpler than that of Āryabhaṭa; hence they must be independent of each other.

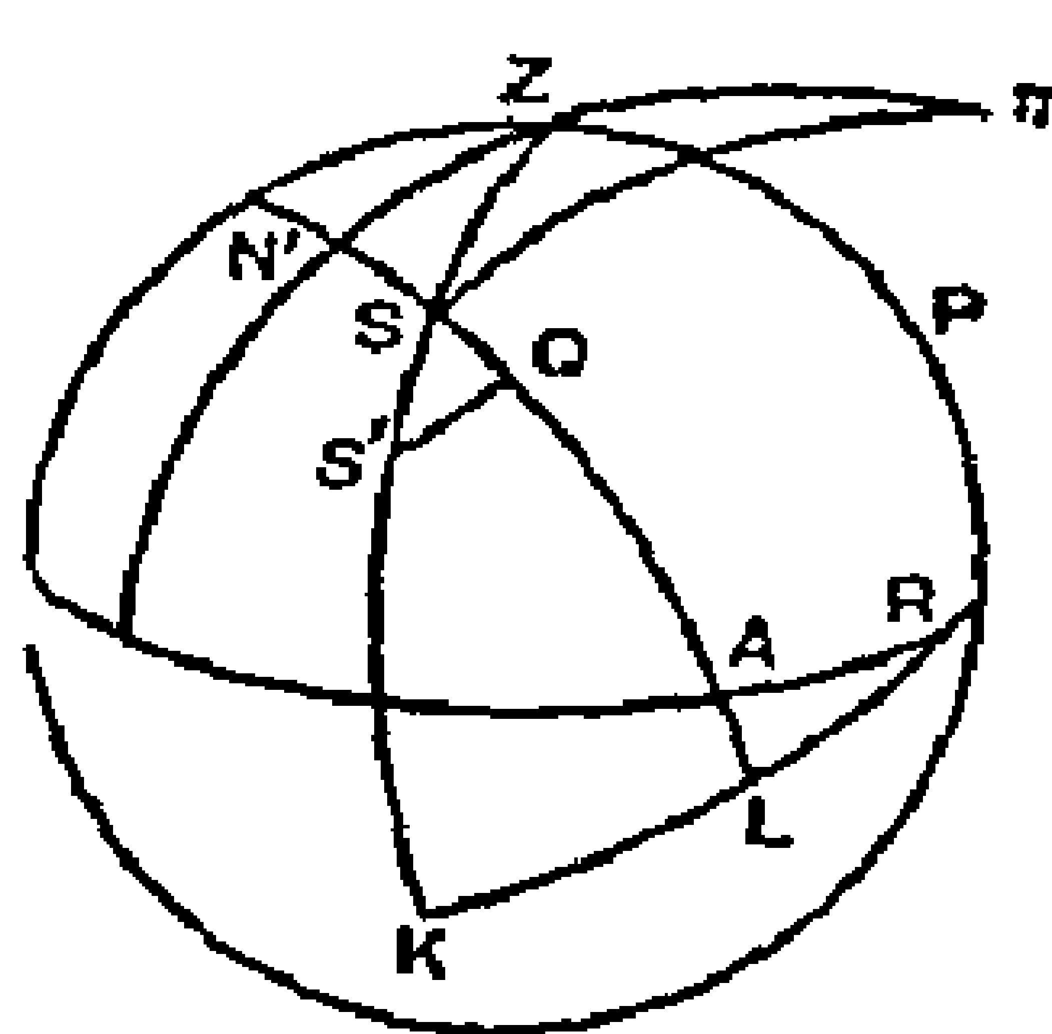


Fig. 12

Problem VII:—

To find the Angle made by the Vertical through any Point of the Ecliptic with the Latter

This problem is considered by Ptolemy but it is not considered separately in Indian Astronomy, but from the rule for parallax in longitude, the rule for its calculation can be deduced.

Indian Method:

In Fig 12 S represents the true position of the Sun and S′ the Sun's position as depressed by parallax. N′SA is the ecliptic. If from S′, S′Q be drawn perpendicular to the ecliptic, then, if P is the horizontal parallax.

1. *Grahagaṇita*, XII, 3-4.

2. Manitius, *ibid*, pp. 110-111.

$$SQ = SS' \times \frac{R\cos S'SQ}{R} = \frac{P \times R\sin ZS}{R} \times \frac{R\cos SSQ}{R}$$

[1] $$= \frac{P}{R}\sqrt{(R\sin ZS)^2 - (R\sin ZN')^2}$$

[2] $$= \frac{P}{R^2} \times R\sin N'S \times R\cos ZN',$$ where N' is the nonagesimal

Thus $R\cos S'SQ$ is seen to be

$$= \frac{R\sin NS \times R\cos ZN'}{R\sin ZS}.$$

The Indian method is fully described by Bhāskara in his [1]*Goladhyaya* VIII 12 25 The truth of the Indian rule for R cos S SQ is easily seen from the spherical triangle πZS where π is the pole of the ecliptic.

Greek Method

[3]Ptolemy takes SK and SL=90° each along the vertical circle ZSEK and the ecliptic N'SA The great circle through K and L cuts the horizon at R which is the pole of the vertical circle. He takes SKL for the triangle and EAR for the transversal, then

$$\frac{\sin SE}{\sin EK} \times \frac{\sin KR}{\sin LR} \times \frac{\sin LA}{\sin AS} = 1$$

or $$\sin LR = \frac{\cos ZS \times \cos AS}{\sin ZS \times \sin AS}$$

or $$\cos S'SQ = \cot ZS \times \cot AS = \tan SE \times \cot AS.$$

The Indian and the Greek rules are altogether different both in form and method There can therefore, be no question of any connection between them

Problem VIII —

To convert the Celestial Longitude of a Heavenly Body into its Polar Longitude

If σ be the position of a (Fig.13) ΥK and σK are the celestial longitude and the celestial latitude respectively. ΥM and σM are the polar longitude and polar latitude. ΥN and σV are the right ascension and declination of the star

Indian Method

All Indian astronomers attempt at finding MK which sub-

1 Āryabhaṭa Gola 34 *Pañcasiddhāntikā* IX 22 BrSpSi XI 23.
2. BrSpSi V 4-5 *Sūryasiddhānta* V 7-9 Bhāskara, *Grahagaṇita* XII 4.
3. Manitius, ibid p. 117

tracted from or added to γK the celestial longitude gives γM the polar longitude

According to Āryabhaṭa[1],

$$MK = \frac{\sigma K \times R \text{ vers } \gamma K \times R \sin \omega}{R^2}.$$

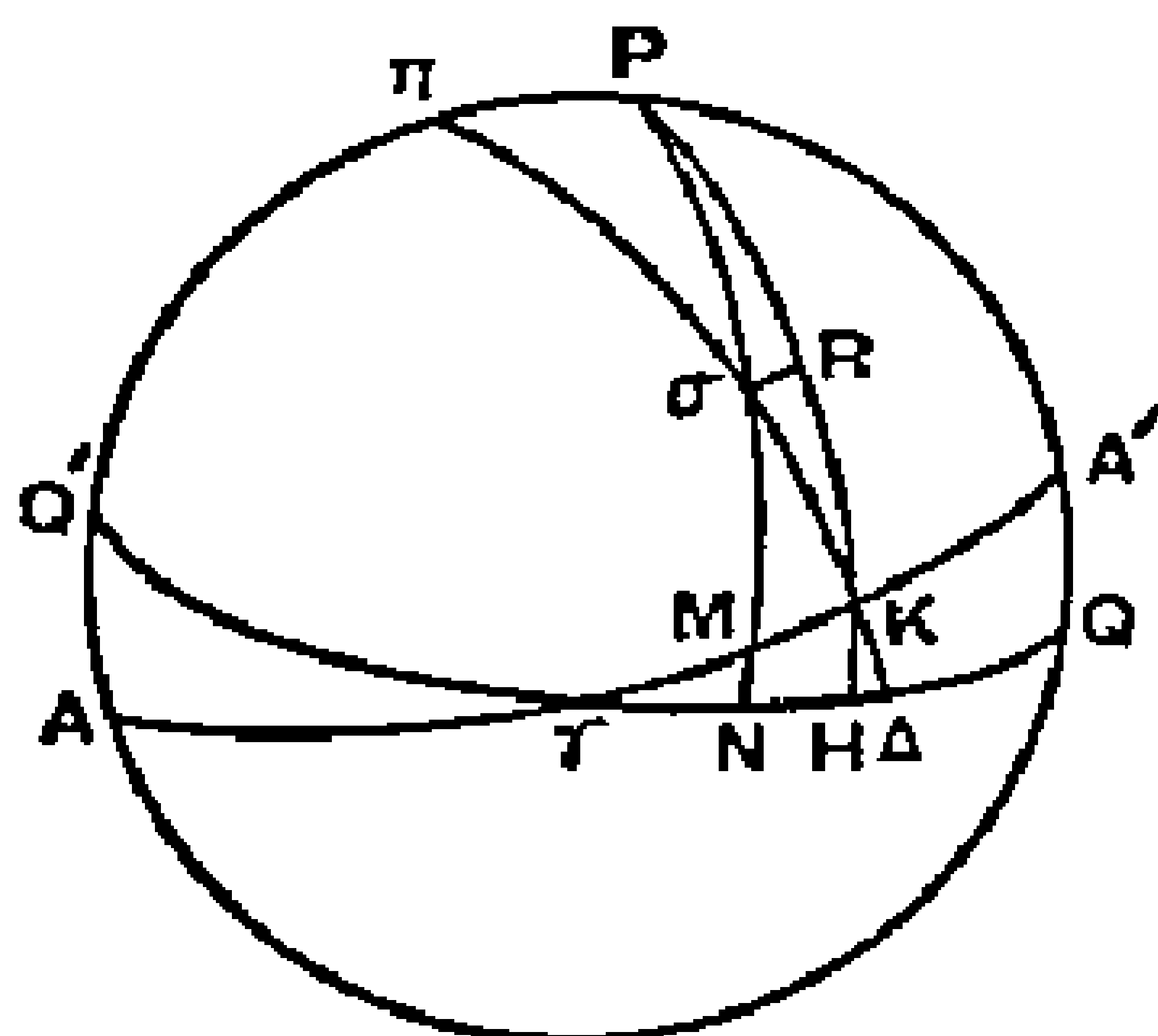


Fig. 13

Brahmagupta[2] makes a distinct improvement on Āryabhaṭa and gives his rule for finding the pro ection MK on the celestial equator

If P be the celestial pole PKH the secondary to the equator Brahmagupta says that

$$NH = \frac{\sigma K \times R \sin (\gamma K + 90°) \times R \sin \omega}{R}$$

If from σ σR is drawn perpendicular to PKH it is evident that

$$R \sin \sigma R = \frac{R \sin \sigma K \times R \sin \sigma KR}{R}$$

According to Āryabhaṭa and Brahmagupta as explained before

$$R \sin \sigma KR = \frac{R \sin (\gamma K + 90°) \times R \sin \omega}{R}$$

Hence Brahmagupta intends that

$$NH = \sigma R = \frac{\sigma K \times R \sin \sigma KR}{R}.$$

which is rather a big assumption He then directs the finding of the part of the ecliptic of which σR or NH is the projection on the equator thus approximately to MK

Āryabhaṭa Brahmagupta[3] and the modern *Sūryasiddhānta* take the declination $\sigma N = \sigma K + KH$ where σK is small They do not consider the case where σK is large

Bhāskara alone gives us fairly correct rules for this transformation of co-ordinates

1 Āryabhaṭa *Gola* 36.
2 *BrSpSi* X 17
3 *BrSpS* X 15 *Sūryasiddhānta* II 58

In order to find σN, he would multiply

σK by $\dfrac{R \cos \sigma KP}{R}$; according to him,

$$\sigma N = \frac{\sigma K \times R \cos \sigma KP}{R} + KH^{1}$$

This is a decided improvement on Brahmagupta's corresponding rule. The declination σN obtained would be very nearly accurate.

Having obtained σN, Bhāskara[2] then directs the finding of NH, thus,

$$NH = \frac{\sigma K \times R \sin \sigma KP}{R \cos \sigma N}$$

He then directs the finding of MK on the ecliptic of which NH is the projection by means of the times of rising of the signs of the zodiac on the equator.

Thus the Indian methods show a beginning and development only. The Greek method as given by Ptolemy is mathematically accurate.

Greek Method[3]

To transform the celestial longitude and celestial latitude to right ascension and declination

Let the great circle $\pi\sigma K$ meet the equator at Δ. Ptolemy would then from the given value of γK, find $\gamma\Delta$ and ΔK[1] by using his tables for the rising of signs of the zodiac on the equator. He then takes $\pi P\sigma$ for the triangle and $\gamma N\Delta Q$ for the transversal. The Menelaus' Equation, then, is

$$\frac{\sin \pi Q}{\sin QP} \times \frac{\sin PN}{\sin N\sigma} \times \frac{\sin \sigma\Delta}{\sin \Delta\pi} = 1^{2}$$

Here $\pi Q = 90^\circ + \omega$, $QP = 90^\circ$, $PN = 90^\circ$, $\sigma\Delta = \sigma K + K\Delta$

$\pi\Delta = 90^\circ + K\Delta$, whence $N\sigma$ is obtained.

He next takes PNQ for the triangle and $\pi\sigma\Delta$ for the transversal.

$$\therefore \frac{\sin P\pi}{\sin \pi Q} \times \frac{\sin Q\Delta}{\sin \Delta N} \times \frac{\sin N\sigma}{\sin \sigma P} = 1$$

Here $P\pi = \omega$, $\pi Q = 90^\circ + \omega$, $Q\Delta = 90^\circ - \gamma\Delta$ Now,

Hence the above equation gives him ΔN

$\gamma N = \gamma\Delta - \Delta N$

1. Bhāskara, Grahagaṇita XIII. 3.
2. Ibid. XIII, 4.
3. Manitius, Ibid. Vol. II. Achtes Buch pp 44-53

tracted from or added to, γK the celestial longitude gives γM the polar longitude

According to Āryabhaṭa[1],

$$MK = \frac{\sigma K \times R \text{ vers } \gamma K \times R \sin \omega}{R^2}.$$

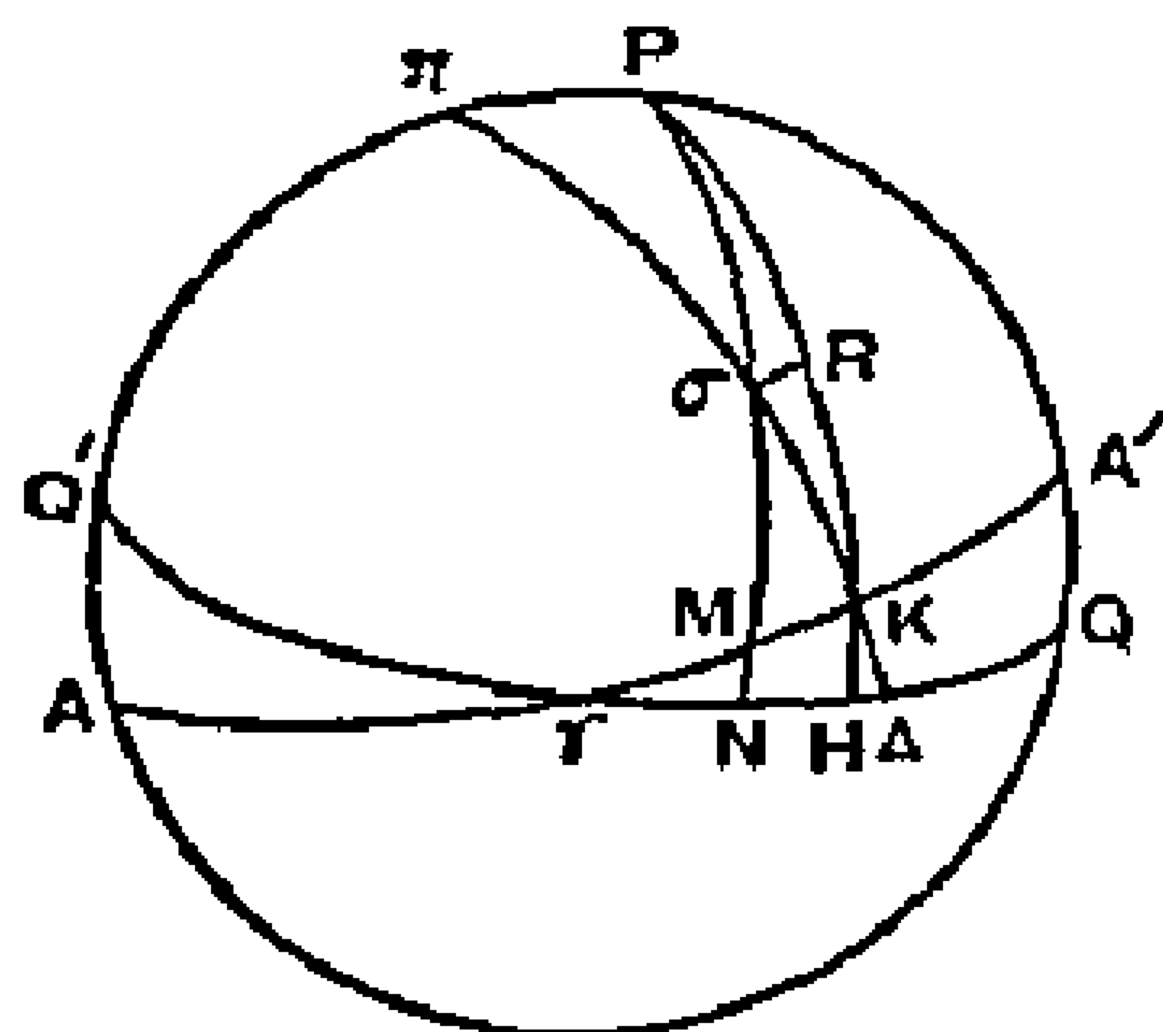


Fig- 13

Brahmagupta[2] makes a distinct improvement on Āryabhaṭa and gives his rule for finding the projection MK on the celestial equator

If P be the celestial pole, PKH the secondary to the equator, Brahmagupta says that

$$NH = \frac{\sigma K \times R \sin (\gamma K + 90°) \times R \sin \omega}{R}$$

If from σ, σR is drawn perpendicular to PKH, it is evident that,

$$R \sin \sigma R = \frac{R \sin \sigma K \times R \sin \sigma KR}{R}$$

According to Āryabhaṭa and Brahmagupta as explained before

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σK by $\frac{R \cos \sigma KP}{R}$; according to him.

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$$NH = \frac{\sigma K \times R \sin \sigma KP}{R \cos \sigma N}$$

He then directs the finding of MK on the ecliptic of which NH is the projection by means of the times of rising of the signs of the zodiac on the equator

Thus the Indian methods show a beginning and development only The Greek method as given by Ptolemy is mathematically accurate.

Greek Method[3]

To transform the celestial longitude and celestial latitude to right ascension and declination

Let the great circle $\pi\sigma K$ meet the equator at Δ. Ptolemy would then from the given value of ΥK, find $\Upsilon\Delta$ and ΔK by using his tables for the rising of signs of the zodiac on the equator. He then takes $\pi P\sigma$ for the triangle and $\Upsilon N\Delta Q$ for the transversal. The Menelaus' Equation then, is

$$\frac{\sin \pi Q}{\sin QP} \times \frac{\sin PN}{\sin N\sigma} \times \frac{\sin \sigma\Delta}{\sin \Delta\pi} = 1$$

Here $\pi Q = 90° + \omega$, $QP = 90°$, $PN = 90°$, $\sigma\Delta = \sigma K + K\Delta$

$\pi\Delta = 90° + K\Delta$, whence $N\sigma$ is obtained

He next takes PNQ for the triangle and $\pi\sigma\Delta$ for the transversal.

$$\therefore \frac{\sin P\pi}{\sin \pi Q} \times \frac{\sin Q\Delta}{\sin \Delta N} \times \frac{\sin N\sigma}{\sin \sigma P} = 1$$

Here $P\pi = \omega$, $\pi Q = 90° + \omega$, $Q\Delta = 90° - \Upsilon\Delta$

Hence the above equation gives him ΔN Now,

$\Upsilon N = \Upsilon\Delta - \Delta N$

1. Bhāskara, *Grahaganita* XIII, 3.
2. Ibid. XIII, 4.
3. Manitius, Ibid. Vol II, *Achtes Buch*, pp 84-85

Let E represent the centre of the Earth (Fig 15) APM the Sun's circular orbit or concentric, let A and P be the apogee and the perigee respectively From EA cut off EC equal to the radius of the Sun's epicycle With centre C and radius equal to EA describe the eccentric A'P S cutting AP and AP produced at P and A' Here A' and P' are the real apogee and perigee of the Sun's orbit Let PM and P S be any two equal arcs measured from P and P'

The idea is that the mean planet M and the apparent Sun S move simultaneously from P and P' in the counterclockwise direction along the concentric and the eccentric circles They move with the same angular motion and arrive simultaneously at M and S

Here EM and CS are parallel and equal hence MS is also equal and parallel to EC Let SH be drawn perpendicular to EM

The angle PEM is the mean anomaly and the angle P'ES the true anomaly, the angle SEM is the equation of the centre is readily seen to be plus (+) from P to A' and minus (−) from A' to P'. *Thus as regards the character of the equation the eccentric circle* is quite right We now turn to exmine how far it is true as to the amount

Let the angle SEM denoted by E and the angle ∠PEM =∠P'CS=θ, EP=CP'=a EC=MS=p then

$$\tan E=\frac{SH}{HE}=\frac{p \sin \theta}{a-p \cos \theta}$$

$$E=\frac{p}{a} \sin \theta-\frac{p^2}{2a^2}\sin 2\theta+\frac{p^3}{3a^3}\sin 3\theta \;\ldots$$

Now the true value of E in elliptic motion is given by

$$E=\left(2e-\frac{e^3}{4}\right)\sin \theta+\frac{5}{4}e^2\sin 2\theta+\frac{13e^3}{12}\sin 3\theta^*, \ldots$$

If we now put $\frac{p}{a}=2e-\frac{e^3}{4}$, *as a first approximation* $\frac{p}{a}=2e$

Hence $\frac{p^2}{2a^2}=2e^2$ which is greater than $\frac{5}{4}e^2$ by $\frac{3}{4}e^2$ In the case of the Sun if the value of p be correctly taken the error in the coefficient of the second term becomes +3, similarly in the case of the Moon the corresponding error becomes +8'

*Godfray's Astronomy p 149

Again if $\frac{p}{a}=2e$, what is the centre of the eccentric circle is the empty focus of the ellipse or that the ancient astronomers practically took the planets to be moving with uniform angular motion round the empty focus This was not a bad approximation

Also ES$=r=$EH approximately

$$r=a\left(1-\frac{p}{a}\cos\theta\right)$$

but in the elliptic motion

$$r=a\,(1-e\cos\theta)\ *$$

Hence the error is not very considerable here also

This is the way in which the ancient astronomers both Greek and Hindu, sought to explain the inequalities in the motion of the Sun and the Moon In the case of the Moon these astronomers took the coefficient $2e-\frac{e^3}{4}=300$ nearly, the modern value is 377 nearly The reason for this has been pointed out to be that the Moon was observed correctly only at times of eclipses At the eclipses of syzygies, the evection term of the Moon s equation diminishes (numerically) the principal ecliptic term by about 76'

We have thus far explained the idea of planetary motion of the ancients under the eccentric circle costruction The same, however, is explained under the epicyclic construction

Let AMP be the circular orbit of the Sun, having E the centre of the earth for the centre (Fig 16) Let the diameter AEP be the apse line A the apogee and P the perigee Let M be the mean position of the Sun in the orbit. With M as the centre describe the epicycle UNS Let EM cut the epicycle at N and U Now the construction for finding S the apparent Sun is thus given.—

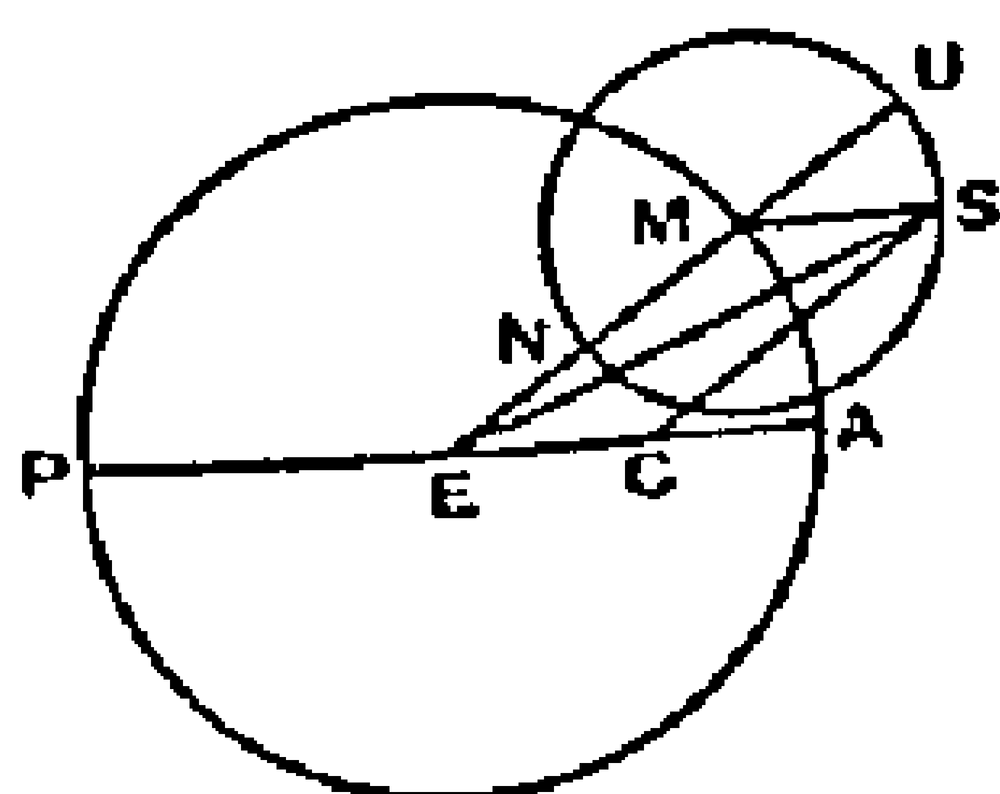


Fig 16

Make ∠UMS= ∠MEA the arc US is measured clockwise whereas the arc A to M is measured counterclockwise

From this construction MS is parallel to EA If EC be measured equal to MS the radius of the epicycle, along EA to-

* Godfray s Astronomy p 143

wards the apogee, then CS is a constant length and C is a fixed point. Hence the locus of S is an equal circle with the centre at C. Thus both the eccentric, and the epicycle and the concentric combined, led to the same position.

It was thus usual to explain the planetary motion under both the assumed constructions; and both gave the position for a planet. The eccentric circle construction appears to be the earlier in the history of astronomy and the latter was later. If the former construction can be traced to Apollonius of Perga who did so much to develop the "conic sections" as science, the reason why he preferred the eccentric circle to the ellipse, appears to be that either that this planetary construction was always deep rooted in the minds of men or that he was carried by the idea that "the circle was the most perfect curve." We are inclined to the view that the eccentric circle idea was transmitted from Babylonia to Greece. We now pass on to consider the Indian construction for the possition of superior and inferior planets.

Superior Planets

With regard to the five planets, Mercury, Venus, Mars, Jupiter and Saturn, the Indian astronomers give only one construction for finding the apparent geocentric position. Each of these "star planets" is conceived as having twofold planetary inequalities: (i) the inequality of apsis, (ii) the inequality of the *śighra*. With regard to the superior planets, the *śighra* apogee or the *śighrocca* coincided with the mean position of the Sun. As Varāhamihira observed, of the other planets beginning with Mars, the Sun is the so-called *śighra*. (*PSi*. XVII. 1)

Let AMSP be the concentric of which the centre E is the same as that of the earth (Fig. 17). A'M₁P' the eccentric circle of apsis of a superior planet, of which the centre is C; A, M, S, P be respectively the apogee, the Mars planet, the direction of the *śighra* and the perigee of the concentric; A', M₁, P' be the apogee, the planet as corrected by the equation of apsis, P' the perigee in the eccentric. The arc

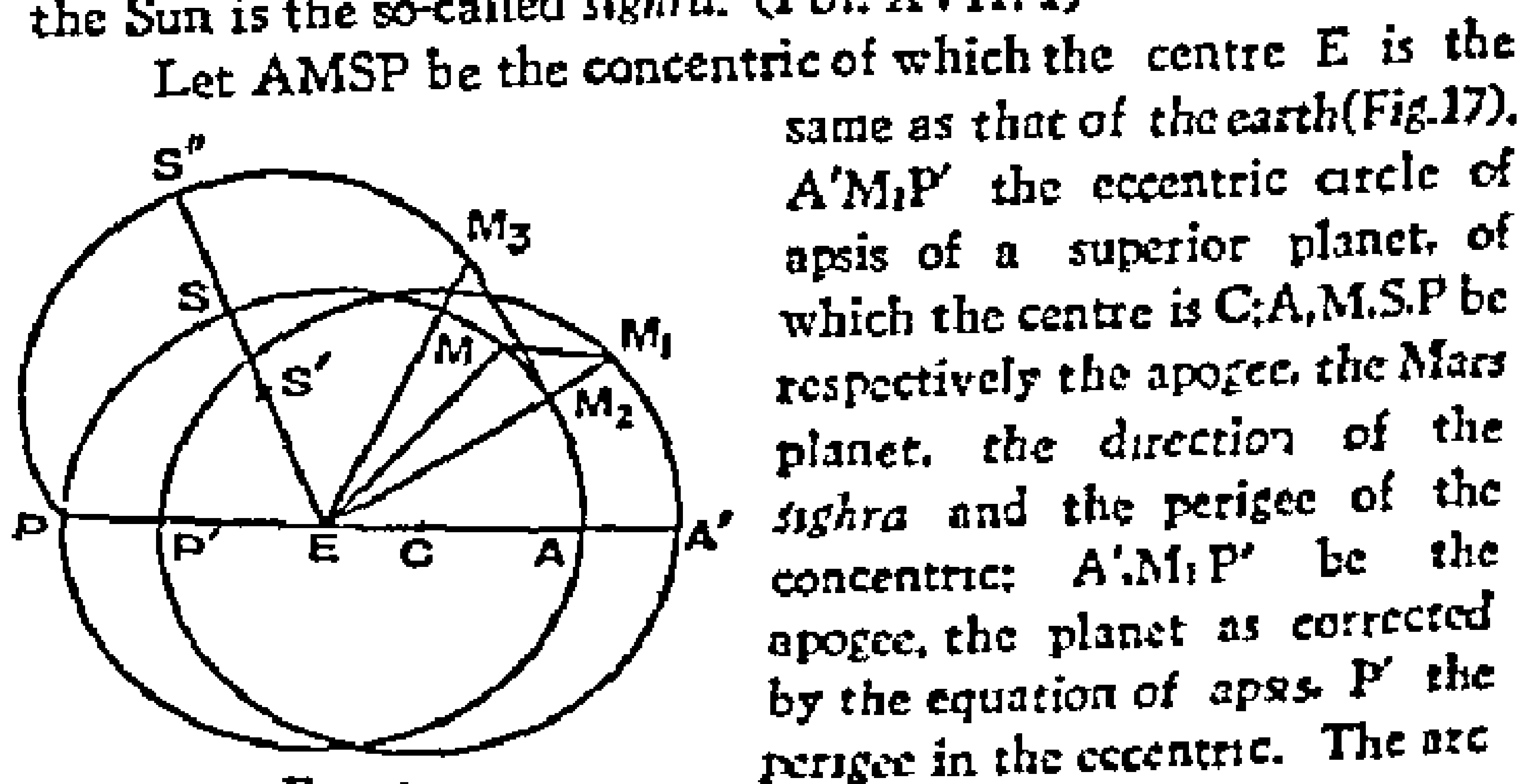


Fig. 17

which is faultless[1]

> This science of *dhūlikarma* has not been imparted by great teachers for blasphemy One who would be using it for this purpose would lose all good name[2]

Brahmagupta uses the term *ganita* only for those calculations which are of arithmetical in nature The science of algebra the foundations of which was laid by Āryabhaṭa I, was named as *kuṭṭaka* or *kuṭṭākāra* by Āryabhaṭa and in the *Brāhmasphuṭa-siddhānta* also it is separately dealt with under *Kuṭṭādhyāya* or *kuṭṭakādhyāya* (Chapter XVIII) Later on the term *bijagaṇita* was specifically given to the science of algebra

The *Kuṭṭādhyāya* of the *Brāhmasphuṭasiddhānta* deals with the (i) concept of *kuṭṭaka* (pulveriser), addition of positive and negative as well as zero quantities, equations in one unknown (*eka-varṇa samikaraṇa*) equations in several unknowns (*aneka varṇa samikaraṇa*), equations involving products of unknowns (*bhāvita*) and quadratic equations (*varga prakṛtih*) (Chapter XVIII of the *Brāhmasphuṭasiddhānta*)

Āryabhata, Bhaskara and Brahmagupta use Place Value Notations

In Europe the first definite traces of the place-value numerals are found in the tenth and eleventh centuries but the numerals came into general use in mathematical text books only in the seventeenth century In India, however Āryabhaṭa I (499) Bhāskara I (522) Lalla (c. 598) and Brahmagupta (628) all use the place value numerals There is no trace of any other system in their works Perhaps in this country we had the place value system as early as 200 B C if not earlier The use of a symbol for zero is found in Piṅgala's *Chandaḥ Sūtra* (perhaps of 200 B C) In literature we have an indication of the place value from about *100 B C and later in the Purānas from the* second to the fourth century A D The *Bakhaśālī Manuscript* (perhaps of 200 A D) uses the place-value notations The earliest use of the place value principle with the letter numerals

1 ग्रहयोगेन ग्रहयुतिरार्यास्त्रिंशन्निपुणाष्टसप्तत्या ।
आद्यार्थे राशिधूलिकर्म योच्चैर्विना करणे ॥ —*BrSpSi X 66*

2 गुरुणा न धूलिकर्म प्रतिकञ्चुककारिणे प्रदातव्यम् ।
दत्त गुरुत्वणाशं कुरुते प्रतिकञ्चुकं यस्य ॥ —*BrSpSi X 67*

hence it means the science of calculation which requires the uses of writing material (the board) The word pāṭi is not Sanskrit (it originated in the non Sanskrit literature in India) the oldest term in Sanskrit for the board is *Phalaka* or *paṭṭa* However this term got currency in the Sanskrit literature also about the beginning of the seventh century Brahmagupta does not use the term *pāṭigaṇita* he favours the use of the term *dhūlikarma* or writing figures on dust spread on a board or on the ground The word *pāṭīgaṇita* was translated into Arabic as *ilm hisab al takht* (calculation on board) and the word *dhūlikarm a* as *hisab al ghobār* (calculation on dust)

Brahmagupta, in the very first verse in the Chapter XII (*Gaṇitādhyāya*) refers to twenty operations (*parikarma*) and eight determinations

> He who distinctly and severally knows the twenty logistics, addition etc and the eight determinations (*vyavahāra*) inculding (measurement by) shadow is a *gaṇaka* (mathematician) [1]

The commentators have given the list of these logistics (*parikarma*) and determinations (*vyavahāra*) as follows,

(A) *Parikarma* or logistics

1. Saṁkalitam (addition)
2. Vyavakalitam (subtraction)
3. Gananam (multiplication)
4. Bhāgahārah (division)
5. Vargah (square)
6. Vargamūlam (square-root)
7. Ghanaḥ (cube)
8. Ghanamūlam (cube root)
9. 13 Five standard forms of fractions (Pañca- āti)
14. Trairāśikam (the rule of three)
15. Vysta trairāśikam (the inverse rule of three)
16. Pañca rāśikam (the rule of five)
17. Sapta-rāśikam (the rule of seven)
18. Nava rāśikam (the rule of n ine)

1 परिकर्म विंशति य संकलिताद्या पृथक् विजानाति ।
अष्टौ च व्यवहारान् छायान्तान् भवति गणक स ॥ —BrSpSi XII 1

19 Ekādaśa rāśikam (the rule of eleven)
20 Bhāṇḍa pratibhāṇḍam (barter and exchange)

(B) *Vyavahāra* or determinations

1 Miśrakah (mixture)
2 Śreḍhī (progession or series)
3 Kṣetram (plane figures)
4 Khātam (excavation)
5 Citih (stock)
6 Krākacikah (saw)
7 Rāśiḥ (*mound*)
8 Chāyā (shadow)

also occur in the Bakhaśāli Manuscript.

Āryabhaṭa I does not mention the everyday methods of multiplication in his *Āryabhaṭiya* probably because they were too elementary to be included in a Siddhānta work Brahmagupta however in a supplement to the section on mathematics in his *Siddhānta* gives the names of some methods with very brief descriptions of the processes —

> The multiplicand repeated as in *gomūtrikā* as often as there are digits in the multiplier is severally multiplied by them and (the results) added according to places this gives the product Or the multiplicand is repeated as many times as there are component parts in the multiplier [1]

(the word *bheda* occurring in the verse has been translated as integrant portions by Colebrooke p 319 Again by the term *bheda* are meant portions which added together make the whole or aliquot parts which multiplied together make the entire quantity

> The multiplicand is multiplied by the sum or the difference of the multiplier and an assumed quantity and from the result the product of the assumed quantity and the multiplicand is subtracted or added [2]

(Colebrooke thinks that this is a method to obtain the true product when the multiplier has been taken to be too great or too small by mistake [3] Datta and Singh think however that this is not correct [4]

Thus Brahmagupta mentions four methods of multiplication (i) *gomūtrikā* (ii) *khaṇḍa* (iii) *bheda* and (iv) *iṣṭa* The common and the well known method of *kapāṭa sandhi* has been omitted by him

---

1 गुणकारखण्डतुल्यो गुण्यो गोमूत्रिकागतो गुणित ।
सहित स्थगुणनो गुणकारभेदतुल्यो वा ॥ —*BrSpSi XII 55*

2 गुण्यो राशिर्गुणकार[illegible]धिकोनेन [illegible] ।
गुण्य[illegible] न युतो गुणितोऽभ्यधिकोनेन [illegible] ॥ —*BrSpSi XII 56*

3 Colebrooke T H., *Hindu Algebra* p 320

4 Datta B. and Singh A N., *History of Hindu Mathematics* Pt. I (Arithmetic) p 135 (1962)

(i) *Gomūtrikā* method or zig zag method The word *gomūtrikā* means 'similar to the course of cow's urine", hence "zigzag' This method in all essentials is the same as the *sthāna khaṇḍa* method The following illustration is based on the commentary of Pṛthūdaka Svāmi

Example To multiply 1223 by 235

The numbers are written thus

```
2    1223
3     1223
5      1223
```

The first line of figures is then multiplied by 2 the process beginning at units place thus 2×3=6 3 is rubbed out and 6 substituted in its place and so on After all the horizontal lines have been multiplied by the corresponding numbers on the left in the vertical line the numbers on the *pāṭī* stand thus

```
2446
 3669
  6115
------
287405
```

after being added together as in the present method

The *sthāna khaṇḍa* and the *gomūtrikā* methods resemble modern plan of multiplication most closely

(ii) *Khaṇḍa* Method or Parts Multiplication Method Since the days of Brahmagupta, this method of multiplication also became very popular We have two methods under this head

(i) The multiplier is broken up into two or more parts whose sum is equal to it The multiplicand is then multiplied severally by these and the results added

To take an example

$$13\times158=(6+7)\times158=(6\times158)+(7\times158)$$
$$=948+1106$$
$$=2054$$

(ii) The multiplier is broken up into two or more aliquot parts. The multiplicand is then multiplied by one of these, the resulting product by the second and so on till all the parts are exhausted The ultimate product is the result.

Thus for example :

$$96\times237=(4\times4\times6)\times237$$
$$=(4\times237)\times4\times6=948\times4\times6$$
$$=(4\times948)\times6=3792\times6$$
$$=22752$$

These methods of multiplication are found among the Arabs and the Italians having obtained from people of India. They were known as the "Scapezzo" and "Repiego" methods respectively amongst Italians.

(iii) *Iṣṭa-guṇana* Method or the Algebraic Method.

We have already quoted the relevant verse from the *Brāhmasphuṭa-siddhānta* in this connection; (XII. 56) :

> The multiplicand is multiplied by the sum or the difference of the multiplied and an assumed quantity and from the result the product of the assvmed quantity and the multiplicand is subtracted or added.[1]

This method is of two kinds according as we (i) add or (b) subtract an assumed number. The assumed number is so chosen as to give two numbers with which multiplication will be easier than with the original multiplier. The two ways are illustrated below :

(i) $93\times13=(93+7)\times13-7\times13=1300-91=1209.$

(ii) $93\times13=(90+3)13=90\times13+3\times13=1170+39$
$=1209$

This method was in use among the Arabs and in Europe, obviously having gone out from this country.

This process has been regarded as an inverse of multiplication The terms used for this operation are *bhāgahāra, bhājana, haraṇa, chedana*, etc., all these terms more or less carrying the sense "to break into parts", "to divide" etc., excepting "*haraṇa*" which denotes "to take away". This term shows the relation of division to the operation of subtraction. The dividend is termed as *bhājya, hārya* etc., the divisor is known as

1. [illegible] ।
[illegible] ॥ —*BrSpSi*. *XII*. 56

bhājaka bhāgahāra or simply *hara* quotient is known as *labdhi* or *labdha* (or what is obtained)

India never regarded this operation as a difficult one, in Europe this operation was regarded as a tedious one till the 15th century or so Division was such a common operation that Āryabhaṭa did not regard it as worth being included in his treatise But since he has given the methods of extracting square-roots and cube-roots which obviously depend on division, we conclude that the method of division was known to him Most *Siddhānta* writers have followed Āryabhaṭa I in omitting this operation from their texts this being regarded too elemantary to be included Brahmagupta does not give details of this operation The later treatises on Arithmetic as Śrīdhara's *Triśatikā* and the *Pāṭīgaṇita* (I 20) and Āryabhaṭa II (c 950 A D) have given the details of this operation

Square

The Sanskrit term for square is *varga* or *kṛti* (*varga* literally means rows' or troops of similar things) In mathematics, it usually means the square power and also the square figure or its area Thus we find in the *Āryabhaṭīya*

> A square figure of four equal sides (and the number representing its area) are called *varga* The product of the two equal quantities is also *varga*[1]

The term *kṛti* means 'doing', making" or "action . It carries with it the idea of specific performance probably the graphical representation

For the first time we have a definite rule for squaring in the writings of Brahmagupta But it does not mean that prior to him it was not known It must have been known to Āryabhaṭa I since he has given the square-root method

Brahmagupta gives his method of squaring briefly as follows

> Combining the product twice the digit in the less (lowest) place into the several others (digits) with its (i e of the digit in the lowest place) square (repeatedly) gives the square[2]

1 [illegible] —*Ārya II* 3

2. [illegible] —*BrSPSi XII* 63

The method has been more clearly enunciated by Mahāvira (850 A D) in the *Gaṇitasārasaṁgraha*

> Having squared the last (digit) multiply the rest by the digits by twice the last (which) is moved forward (by one place) Then moving the remaining digits continue the same operation (process) This gives the square[1]

Brahmagupta s method of squaring is shown by the following example

To square 125

The number is written down

125

The square of the digit in the last place i e $5^2=25$ is set over it thus

25
125

Then $2 \times 5=10$ is placed below the other digits and 5 is rubbed out thus

25
12
10

Multiplying by 10 the rest of the digits i e 12 and setting the product over them (the digits) we have

1225
12
10

Then rubbing out 10 which is not required and moving the rest of the digits i. e 12 we have

1225
12

Thus one round of operations is completed

Again as before setting the square of 2 above it and $2 \times 2=4$ below 1 we have

1625
1
4

---

1 *GSS P 12*

Multiplying the remaining digit 1 by 4 and setting the product above it we have

5625
1

Then moving the remaining digit 1 we obtain

5625
1

Thus the second round of operations is completed

Next setting the square of 1 above it the process is completed for there are no remaining figures and the result stands thus

15625

Algebraic Method of Squaring

Brahmagupta in his *Brāhmasphuṭasiddhānta* gives a minor method of squaring thus

> The product of the sum and the difference of the number (to be squared) and an assumed number plus the square of the assumed number give square[1]

This may be represented by the following identity

$$n^2=(n-a)\,(n+a)+a^2$$

This identity has been used for squaring by most of the Indian mathematicians Thus

$$15^2=(15-5)\,(15+5)+5^2=225$$

We are not giving here other identities which have been used by latter mathematicians of India in getting the squares of numbers for example when *Mahāvīra* says

> The sum of the squares of the two or more portions of the number together with their products each with the others multiplied by two gives the square[2]

he obviously refers to the identity

$$(a+b+c \qquad)^2=a^2+b^2+c^2+ \qquad +2ab+$$

bhaṭṭya

> The continued product of three equals and also the solid having twelve (equal) edges are called *ghana*[1]

A method of cubing applicable to numbers written in the decimal place value notation has been in use in this country from before the 5th century A D Āryabhata I (499 A D) had the familiarity with this method he however does not give the method of cubing in his treatise though he describes the inverse process of extracting the cube root

Brahmagupta gives the method of cubing in the following verse

> Set down the cube of the last (*antya*) then place at the next place from it thrice the square of the last multiplied by the succeeding then place at the next place thrice the square of the succeeding multiplied by the last and (at the next place) the cube of the succeeding This gives the cube[2]

The rule may be illustrated by an example

Example To cube 1357

The given number has four places i e four portions First we take the last digit 1 and the succeeding digit 3 i e 13 and apply the method of cubing thus

(i) Cube of the last (1^3)	=	1
(ii) Thrice the square of the last ($3 \cdot 1^2$) multiplied by the succeeding (3) gives ($3 \cdot 3 \cdot 1^2$)	=	9 (placing at the next place)
(iii) Thrice the square of the succeeding multiplied by the last gives ($3 \cdot 3^2 \cdot 1$)	=	27 (placing at the next place)
(iv) Cube of the succeeding (3^3)	=	27 (placing at the next place)
Thus 13^3 is the sum		2197

1 सदृशत्रयसंवर्गो घनस्तथा द्वादशाश्रिस्स्यात् ॥ —*Ārya II 3*

2 स्थाप्योऽन्त्यघनोऽन्त्यकृतिस्त्रिगुणोत्तरसंगुणा च तत्प्रथमात् ।
उत्तरकृतिरन्त्यगुणा त्रिगुणा चोत्तर घनश्च घन । —*BrSpSi XII 6*

After this we take the next figure 5 i e the number 135 and in this consider 13 as the last and 5 as the succeeding Then the method proceeds thus

(i) The cube of the last (13^3) as already obtained = 2197

(ii) Thrice the square of the last multiplied by the succeeding i e $3\ 13^2\ 5$ = 2535 (placing at the next place)

(iii) Thrice the square of the succeeding multiplied by the last i e $3\ 5^2\ 13$ = 975 (placing at the next place)

(iv) Cube of the succeeding i e 5^3 = 125 (placing at the next place)

Thus 135^3 is the sum 2460375

Now the remaining figure 7 is taken, so that the number is 1357 of which 135 is the last and 7 the succeeding The method proceeds thus

(i) Cube of the last i e (135^3) as already obtained = 2460375

(ii) Thrice the square of the last into the succeeding i e $3\ (135)^2\ 7$ = 382725 (placing at the next place)

(iii) Thrice the square of the succeeding into the last, i e $3\ 7^2\ 135$ = 19845 (placing at the next place)

(iv) Cube of the succeeding i e 7^3 = 343 (placing at the next place)

Thus $(1357)^3$ is the sum 2498846293

Evidently these methods of cubing are based on the identity

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

and keeping in mind the place values of numerals [illegible]

number (this accounts for keeping the results of each of the four operations at the next place)

Square-Root

Indian synonyms for square-root are *vargamūla* or *pada* of a *kṛti* The word *mūla* means the 'root' of a tree which may also mean the foot or the lowest part or bottom of a thing and hence pada or foot also became a synonym of root Brahma Gupta defines square root as follows

> The *pada* (root) of a *kṛti* (square) is that of which it is the square[1]

While the word *mūla* for *root* is the oldest in Indian literature (it occurs in *Anuyogadvāra sūtra* c 100 B C) the word *pada* for root probably for the first time occurs in the writings of Brahmagupta The term *mūla* was borrowed by the Arabs who translated it by *jadhr* meaning basis of square The Latin term *radix* also is a translation of the term *mūla* In the Śulba literature and in the Prākrta texts we find a term *karaṇī* for square-root In geometry this term *karaṇī* means a side In later days the term *karaṇī* was reserved for surds i e. a square-root which cannot be exactly evaluated but which may be represented by a line

We would like to quote here a rule for determining square root of numbers from the *Āryabhaṭīya*

> Always divide the even place by twice the square-root (upon the preceding odd place) after having subtracted from the odd place the square (of the quotient) the quotient put down at the next place (in the line of the root) gives the root[2]

As an illustration we shall proceed to find the square-root of 18225

The odd and even places are marked out by vertical (I) and horizontal (—) lines The other steps are as follows

---

1 पदं कृतिर्यत् तत् — *BrSpSi* XVIII 35

2 भागं हरेदवर्गान्नित्यं द्विगुणेन वर्गमूलेन ।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम् ॥ —*Ārya* II 4

	ı – ı – ı 1 8 2 2 5	
Subtract square	1	root = 1
Divide by twice the root	2) 8 (3	placing quotient at the
	6	next place the root=13
	22	
Subtract square of quotient	9	
Divide by twice	26)132(5	placing quotient at the
the root	130	next place the root=135
Subtract square	25	
of the quotient	25	

The process ends The square root of 18225 is thus 135

It has been stated by Kaye that Āryabhata s method of finding out the square-root is algebraic in character and that it resembles the method given by Theon of Alexandria Ārya bhaṭa s method is purely arithmetic and not algebraic is the view of Datta and Singh who do not agree with Kaye on this point

Cube Root

The Sanskr t term for cube-root is *ghanamūla* or *ghanapada* The first mention of the operation of cube-root is found in the *Āryabhaṭ ya* of Āryabhaṭa I (499 A D), though the operation is given in only a concise form

> Divide the second *aghana* place by thrice the square of the cube-root subtract from the first *aghana* place the square of the quotient multiplied by thrice the preceding (cube root) and (subtract) the cube (of the quotient) from the *ghana* place (the quotient put down) at the next place (in the line of the root) gives (the root) [1]

As has been explained by all the commentators on the *Āryabhaṭıya* the units place is *ghana* the tens place is first *aghana* the hundreds place is the second aghana the thousands place is ghana the ten thousands place is first *aghana* the hun

1 अघनाद् भजेद् द्वितीयाद् त्रिगुणेन घनस्य मूलवर्गेण ।
वर्गस्त्रिपूर्व गुणितश्शोध्य प्रथमाद् घनश्च घनाद् ॥ —*Ārya* II 5

dred thousands place is second *aghana* and so on. Thus to find out the cube root one has to mark out the *ghana*, first *aghana* and second *aghana* places, then the process of finding out the cube root begins with the subtraction of the greatest cube number from the figures up to the last *ghana* place. Though this has not been explicitly mentioned in the rule, the commentators say that it is implied in the expression *ghanasya mūla-vargeṇa* etc. (by the square of the cube-root etc.).

We are reproducing here an illustration given by Datta and Singh.

Example: Find the cube-root of 1953125.

The places are divided into groups of three by marking them as below [*ghana* (|) first *aghana* (—) and second *aghana* (—)]

	| — — | — — |	
	1 9 5 3 1 2 5	
Subtract cube	1	(c) Root=1
Divide by thrice square of root i.e. 3.1^2	3)9(2	(a) Placing quotient after the root 1 gives the root 12
Subtract square of quotient multiplied by thrice the previous root i.e. $2^2.3.1$	6 35 12	(b)
Subtract cube of quotient i.e. 2^3	233 8	(c)
Divide by thrice square of the root i.e. 3.12^2	432)2251(5 2160	(a) Placing quotient after the root 12 gives the root 125
Subtract square of quotient multiplied by thrice the previous root i.e. $5^2.3.12$	912 900	(b)
Subtract cube of quotient i.e. 5^3	125 125	(c)

Thus the cube-root=125.

From the details given it would be clear that the present

method of extracting the cube-root is almost a contraction of the method first given by Āryabhaṭa I (499 A D)

The method of Āryabhaṭa has been invariably followed by Indian mathematicians Brahmagupta in his *Brāhmasphuṭa siddhānta repeats the method in the following words*

> The divisor for the second *aghana* place is thrice the square of the cube-root, the square of the quotient multiplied by three and the preceeding (root) must be subtracted from the next (*aghana* place to the right) and the cube (of the quotient) from the *ghana* place (the procedure repeated gives) the root [1]

Śrīdhara and Āryabhaṭa II have further improved on the method of extracting cube root proposed by Āryabhaṭa I and followed by Brahmagupta Rule for finding the cube root as given by Śrīdhara in his *Pāṭīgaṇita* is as follows

> (Divide the digits beginning with the units' place into periods of) one *ghana* pada (one ' cube ' place) and two *aghana padas* (two "non-cube places) Then subtracting the (greatest possible) cube from the (last) *ghana pada* and placing the (cube) root underneath the third place (to the right of the last *ghana-pada*) divide out the remainder up to one place less (than that occupied by the cube root) by thrice the square of the cube-root which, is not destroyed Setting down the quotient (obtained from division) in the line (of the cube root), (and designating the quotient as the 'first (*ādima*) and the cube root as the 'last (*antya*) subtract the square of that quotient, as multiplied by thrice the last' (*antya*) from one place less than that occupied by the quotient (*uparima rāśi*) as before and the cube of the first (*ādima*) from its own place
>
> (The number now standing in the line of cube-root is the cube-root of the given number up to its last but one *ghana-pada* (cube place) from the left)
>
> Again apply the rule, (placing cube root) under the third place etc (provided there be more than two *ghana-padas* (cube places) in the given number and

1 छेदो घनाद् द्वितीयाद् घनमूलकृतिस्त्रिसंगुणाप्तकृतिः ।
शोध्या त्रिपूर्वगुणिता प्रथमाद् घनतो घनो मूलम् ॥ —*BrSpSi* XII 7

continue the process till all *ghana-padas* (cube-places) are exhausted). This will give the (cube) root (of the given number).[1]

K.S Shukla in his translation and commentary of this book has given the illustration of extracting cube - root as follows :

Example :- To find the cube root of 277167808.

Let us indicate *ghana-padas* or 'cube' places by "c" and *aghana-padas* or non-cube places as "n" :

n n c n n c n n c
2 7 7 1 6 7 8 0 8

Subract the greatest possible cube (i.e. 6^3 or 216) from the last 'cube' place (i.e. from 277) and place the cube-root (i e. 6) underneath the third place to the right of the last 'cube' place. thus we have

n n c n n c n n c
6 1 1 6 7 8 0 8 (remainder)
6 (line of cube-root)

Dividing out by thrice the square of the cube-root (i.e. by $3\,6^2$ or 108) the remainder up to one place less than that occupied by the cube-root (i.e. 611) and setting down the quotient in the line of the cube-root (to the right of the cube-root), we have

n n c n n c n n c
7 1 6 7 8 0 8 (remainder)
6 5 (line of cube-root)

Let now quotient 5 be called the 'first' (*adima*) and the cube-root 6 the 'last' (*antya*). Then subtracting the square of the 'first' (*adima*) as multiplied by thrice the 'last' (*antya*) (i e. $3 \times 6 \times 5^2$ or 450) from one place less than that occupied by the quotient (i e. from 716). we get

n n c n n c n n c
2 6 6 7 8 0 8 (remainder)
6 5 (line of cube-root)

And subtracting the cube of the first (*ādima*) (i e 5^3 or 125) from its own place (i e from 2667) we get

n n c n n c n n c
2 5 4 2 8 0 8 (remainder)
6 5 (line of cube-root)

One round of the operation is now over and the number 65 *standing in the line of the cube root is* the cube-root of the given number (277167808) up to its last but one cube place (*ghana pada*) from the left (i e of 277167)

As there is one more 'cube' place (*ghana pada*) on the right the process is repeated Thus placing the cube-root (*i e* 65) *under the third place beginning with the* last but-one cube' place (*ghana pada*) we have

n n c n n c n n c
2 5 4 2 8 0 8 (remainder)
6 5 (line of cube-root)

Dividing out 25428 by $3 \cdot 65^2$ (=12675) as before and placing the quotient *in the line of the cube root*, we have

n n c n n c n n c
7 8 0 8 (remainder)
6 5 2 (line of cube-root)

Subtracting $3 \times 65 \times 2^2$ (=780) we get

n n c n n c n n c
8 (remainder)
6 5 2 (line of cube-root)

Finally subtracting $2^3 = 8$ from 8 we get

n n c n n c n n c
0 (remainder)
6 5 2 (*line of cube-root*)

The second round of operation is now over There being no more of *ghana-pada* (cube' place) on the right, the process ends The quantity in the line of cube root *viz.*, 652, is the cube-root of the given

number. The remainder being zero the cube root is exact.

Fractions

The concept of fractions in India can be traced to very early times In the *Rgveda*,[1] we find such terms as one-half (*ardha*) and three fourths (*tri-pāda*) In a passage of the *Maitrāyaṇī Saṁhitā*[2] are mentioned the fractions one-sixteenth (*kalā*), one-twelfth (*kuṣṭha*), one eighth (*śapha*) and one fourth (*pāda*) In the *Śulba Sūtras*[3] we have not only a mention of fractions but they have been used in the statement and solution of problems of geometric nature Here in the *Śulba*, unit fractions are denoted by the use of cardinal number with the term *bhāga* or *aṁśa*, thus *pañca daśa bhāga* (literally "fifteen parts") is equivalent to one fifteenth, *sapta bhāga* (literally, "seven parts") is equivalent to one seventh, and so on .. The use of ordinal numbers with the term *bhāga* or *aṁśa* is also quite common thus *pañcama bhāga* stands for one-fifth The composite fractions like *tri aṣṭama* stands for three-eighths and *dvi-saptama* for two-sevenths In the *Bakhaśālī Manuscript*, the term *tryaṣṭa* occurs for 3/8 and $3\frac{3}{8}$ is called *trayastrayaṣṭa* (three three-eighths)

The Sanskrit term for fraction is *bhinna* (literally meaning broken') Obviously the European terms as *fractio*, *fraction*, *roupt*, *rotto* or *rocto* are translations of the same term, they are derived from the Latin *fractus* (*frangere*) or *ruptus* meaning 'broken' The Indian term *bhinna* has a few more connotations, it stands for such numbers of the form :

$$\left(\frac{a}{b}\pm\frac{c}{d}\right),\ \left(\frac{a}{b}\text{ of }\frac{c}{d}\right),\ \left(\frac{a}{b}\pm\frac{c}{d}\text{ of }\frac{a}{b}\right)\text{or}\left(a\pm\frac{b}{c}\right)$$

These forms were termed *jāti* i e, 'classes', and the Indian treatises contain special rules for their reduction to proper fractions Śridhara and Mahāvira each enumerate six *jātis* while our author Brahmagupta gives only five (Bhāskara II gives only four) The need for division of fractions in 'classes' arose out of the lack of proper symbolism to indicate mathematical operations (Datta and Singh *Arithmetic*, p 188) The only operational symbol in use was a dot standing for the negative sign

1 *Rv* X 90 4

2 *Mait* S III. 7.7

3 B Datta *Śulba*, pp 212ff

and the form

$$\frac{p}{q}+\frac{r}{s} \text{ of } \frac{p}{q}+\frac{t}{u}\text{ of}\left(\frac{p}{q}+\frac{r}{s}\text{ of}\frac{p}{q}\right)+\ldots\ldots\ldots$$

is written as

$$\begin{array}{|c|}\hline p\\ q\\ \hline r\\ s\\ \hline t\\ u\\ \hline\end{array}$$

(iv) *Bhāgāpavāha* i e, the form $\left(a-\frac{b}{c}\right)$ is written as

$$\begin{array}{|c|}\hline a\\ b\\ c\\ \hline\end{array}$$

and the form $\frac{p}{q}-\frac{r}{s}\text{ of}\frac{p}{q}-\frac{t}{u}\text{of}\left(\frac{p}{q}-\frac{r}{s}\text{of }\frac{p}{q}\right)-\ldots\ldots$

is written as

$$\begin{array}{|c|}\hline p\\ q\\ \hline r\\ s\\ \hline t\\ u\\ \hline\end{array}$$

(v) Bhāga bhāga . The form

$$\left(a-\frac{b}{c}\right)\text{or}\left(\frac{p}{q}-\frac{r}{s}\right)$$

There does not appear to have been any notation for division, such compounds being written as

$$\begin{array}{|c|}\hline a\\ b\\ c\\ \hline\end{array}\text{ or }\begin{array}{|c|}\hline p\\ q\\ \hline r\\ s\\ \hline\end{array}$$

just as for *bhāgānubandha* That division is to be performed was known from the problem, e g, $1\div\frac{1}{6}$ was written as ṣaḍ bhāga-bhāga, i e, "one-sixth *bhāga bhāga*" or "one divided by one-sixth" It is only in the Bakhśālī Manuscript that the term *bha* is sometimes placed before or after the quantity affected

(vi) *Bhāga-mātṛ*, i e, combinations of forms enumerated alone. Mahāvīra, the author of the *Gaṇitasārasaṁgraha* (850

A D) gives twenty six variations of this class We shall illus trate it by the following example from Śrīdhara

> What is the result when half one-fourth of one fourth, one divided by one-third, half plus half of itself and one third diminished by half of itself, are added together ? (*Triśatikā* p 12)

A modern writer would have written it as

$$\frac{1}{2}+(\frac{1}{4} \text{ of } \frac{1}{4})+(1-\frac{1}{3})+(\frac{1}{2}+\frac{1}{2} \text{ of } \frac{1}{2})(\frac{1}{3}-\frac{1}{2} \text{ of } \frac{1}{3})$$

In the old Indian notation it is written as

1	1	1	1	1 1
2	4	4	1	2 3
			3	1 1
				2 2

The defect of the notation is obvious

1	1
4	4

can be read also as $\frac{1}{4}+\frac{1}{4}$ and

1
1
3

can also be read as $1\frac{1}{3}$

And therefore the original meaning is inferred from the context or from the enunciation of the problem

The rules for reduction of the first two classes (*bhāga* and *prabhāga*) are those of addition or subtraction and multiplica tion The rule for the reduction of the third (*bhāganubandha*) and fourth (*bhāgāpavāha*) classes are given by Brahmagupta in the *Brāhmasphuṭa siddhānta* thus

> The (upper) denominator is multiplied by the denominator and the upper numerator by the same (denominator) increased or diminished by its own numera tor [1]

'Numerator is known as *aṁśa* and the "denominator' as '*cheda*

We give here from Śrīdhara s *Pāṭigaṇita* (about 900 A D according to K S Shukla 750 A D according to Datta and Singh) a *rule for reducing a fraction of the bhāgānubandha class* (i e, a whole number increased by a fraction or a fraction increased by a fraction itself)

1 ऊर्ध्वो हारश्छेदगुणस्तदूनयुक्तो द्यो पृथक्पृथग्वो ।
छेदैश्छेदा गुणिता स्वांशयुतोनैकरूपा ॥ —*BrSpSi XII 9*

> In the *bhāgānubandha* class, the whole number (*rūpa gaṇa*) is multiplied by the denominator (of the fraction) should be increased by the numerator (of the fraction) or the upper denominator having been multiplied by the lower denominator the initial numerator (i e the upper numerator) should be multiplied by the sum of the lower numerator and denominator[1]

(*Pāṭīgaṇita*, 39 cf *BrSpSi* XII 9 (ii) *GSS* (iii) 113

This means that

(i) $a+\frac{b}{c}=\frac{ac+b}{c}$

(ii) $\frac{a}{b}+\frac{c}{d}$ of $\frac{a}{b}$ (which was written by Indians in the style

$$\boxed{\begin{array}{c}\frac{a}{b}\\ \frac{c}{d}\end{array}}$$

is equal to $\frac{a(d+c)}{bd}$

Addition and Subtraction of Fractions

In the *Brāhmasphuṭa-siddhānta* Brahmagupta gives the rule for the addition and subtraction of fractions

> If the denominators (*cheda*) of fractions are different then reduce these fractions to a common denominator Now for the additions, unite the numerators and take their difference in case of subtraction[2]

Brahmagupta and Mahāvīra give the method under *Bhāgajāti*

Multiplication

Brahmagupta says

> The product of the numerators divided by the pro-

1 भागानुबन्धजातौ रूपगणश्छेद सङ्गुण सांश ।
भवरहरघ्नोर्ध्वं हरेऽधोंऽशयुतहरघ्न आद्यंश ॥ —*Pāṭīgaṇita* 39

2 विपरीतच्छेदगुणा राश्योश्छेदांशका समच्छेदा ।
संकलितेऽशा योज्या व्यवकलितेऽशान्तरं कार्यम् ॥ —*BrSpSi* XII 2

> duct of the denominators is the (result of) multiplication of two or more fractions.[1]

While all other writers give the rule in the same way as Brahmagupta, Mahāvīra in the *Gaṇitasārasaṁgraha* refers to cross reduction in order to shorten the work :

> In the multiplication of fractions the numerators are to be multiplied by the numerators and the denominators by denominators, after carrying out the process of cross reduction, if that be possible.[2]

Division of Fractions

The *Āryabhaṭīya* does not explicitly give the rule of division, but under the Rule of Three, we have an indication of this operation. The Rule of Three states the result as $\frac{f \times i}{p}$, where f stands for *phala* i.e. "fruit", i for *icchā*, i.e., demand or requisition, and p for *pramāṇa* i.e. argument. When these quantities are fractional, we get an expression of the form

$$\frac{\frac{a}{b} \times \frac{c}{d}}{\frac{m}{n}}$$

for the evaluation of which Āryabhaṭa I states :

> The multipliers and the divisor are multiplied by the denominators of each other.

These quantities are written in the following way

a	m
b	n
c	
d	

Transferring the denominators we have

a	m
n	b
c	d

Performing multiplication, the result is $\frac{anc}{mbd}$. The above interpretation of the obscure line in the *Āryabhaṭīya* is based

---

1 [illegible] ।
[illegible] ॥ —*BrSpSi* XII 3

2. GSS, p 25 (2)

on the commentaries of Sūryadeva and Bhāskara I (the commentary of Parameśvara on this line is vague and misleading). Sūryadeva in this connection says :

> Here by the word *guṇakāra* is meant the multiplier and multiplicand, i.e., the *phala* and *icchā* quantities that are multiplied together. By *Bhāgahāra* is meant the *pramāṇa* quantity. The denominators of the *phala* and *icchā* are taken to the *pramāṇa*. The denominator of the *pramāṇa* is taken with the *phala* and *icchā*. Then multiplying these, i.e., (the numerators of) the *phala* and *icchā* and this denominator, and dividing by (the product of) the numbers standing with the *pramāṇa* the result is the quotient of the fractions.

Brahmagupta gives the method of division as follows :

> The denominator and numerator of the divisor having been interchanged, the denominator of the dividend is multiplied by the (new) numerator. Thus division of *proper fractions* is performed.[1]

Square and Square-Root of Fractions

Brahmagupta says as follows in this connection —

> The square of the numerator of a proper fraction divided by the square of the denominator gives the square[2].

This rule of Brahmagupta has been followed by other authors also. The rule regarding the square-root as given by Brahmagupta is as follows :

> The square-root of the numerator of a proper fraction divided by the square-root of the denominator gives the square-root.[3]

The Rule of Three :

The Indian term in Sanskrit for the Rule of Three is *Trairāśika* (literally, "three terms"). The term occurs in the *Bakhśālī Manuscript* also, and also in the *Āryabhaṭīya*, indicating the

---

1. परिवर्त्य भागहारच्छेदांशौ छेद अनुपच्छेदः ।
   स राशिगुणो भाज्यस्य भागहारः सवर्णगतो ॥ —*BrSpSi*. XII. 4
2. सवर्णवर्गोऽंशवर्गश्छेदकृतिविभाजितो भवति वर्गः । —*BrSpSi*. XII. 5 (1)
3. सवर्णितान्मूलं छेदपदेनोद्धृतं मूलम् । —*BrSpSi*. XII. 5 (2)

antiquity of the term Bhāskara in his commentary of the *Āryabhaṭīya* gives a justification of the use of this term for the Rule of Three thus:

> Here three quantities are needed (in the statement and calculation) so the method is called *trairāśika* (meaning thereby the "rule of three terms")

The problem of the Rule of Three has the form

> If p (*pramāṇa*) yields f (*phala*), what will i (*icchā*) yield?

Āryabhaṭa II (the author of the *Mahāsiddhānta*, 950 A D) uses the terms *māna*, *vinimaya* and *icchā*, instead of *pramāṇa*, *phala* and *icchā* respectively It has also been pointed out by several authors that the first and third terms are similar, i e, of the same denomination

We shall give here the Rule of Three as given by Āryabhaṭa I and Brahmagupta

> In the Rule of Three, the *phala* ("fruit"), being multiplied by the *icchā* ("requisition") is divided by the *pramāṇa* ("argument"). The quotient is the fruit corresponding to the *icchā* The denominators of one being multiplied with the other give the multiplier (i e, numerator) and the divisor (i e denominator)[1]
>
> In the Rule of Three *pramāṇa* ("argument"), *phala* ("fruit") and *icchā* ("requisition") are the (given) terms, the first and the last terms must be similar. The *icchā* multiplied by the *phala* and divided by the *pramāṇa* gives the fruit (of the demand)[2]

Śrīdhara also gives the Rule of Three almost in the same words. Bhāskara II, Nārāyaṇa and others follow Brahmagupta and Śrīdhara in the *Trairāśika* operation Śrīdhara in his *Pāṭīgaṇita* says

---

1 त्रैराशिकफलराशिं तमथेच्छाराशिना हतं कृत्वा ।
लब्धं प्रमाणभजितं तस्मादिच्छाफलमिदं स्यात् ॥
छेदाः परस्परहता भवन्ति गुणकारभागहाराणाम् ।
छेदगुणं सच्छेदं परस्परं तत्सवर्णत्वम् ॥ —*Ārya* II 26-27.

2 त्रैराशिके प्रमाणं फलमिच्छाद्यन्तयोः सदृशराशी ।
इच्छाफलेन गुणिता प्रमाणभक्ता फलं भवति ॥ —*BrSpSi* XII. 10

> In (solving problems on) the Rule of Three, the argument (*pramāṇa*) and the requisition (*icchā*), which are of the same denomination, should be set down in the first and last places; the fruit (*phala*), which is of a different denomination, should be set down in the middle, (this having been done) that (middle quantity multiplied by the last quantity should be divided by the first quantity.[1]

We shall illustrate the Rule of Three by an example from the *Pāṭīgaṇita* (Example 25):

> Example, If 1 *pala* and 1 *karṣa* of sandalwood are obtained for ten and a half *paṇas*, then for how much will nine *palas* and one *karṣa* (of sandalwood) be obtained ?[2]

Here in this Example.

argument=1 *pala* and 1 *karṣa*=$1\frac{1}{4}$ or 5/4 *palas*; fruit=$10\frac{1}{2}$ or 21/2 *paṇas*;

and requisition=9 palas and 1 karṣa=$9\frac{1}{4}$ or 37/4 palas. According to the Rule we shall write them as :

1	10	9
1	1	1
4	2	4

Converting these into proper fractions we have

5	21	37
4	2	4

Then applying the rule, (i.e. multiplying the second and the last and dividing by the first), we have

21	5
2	4
37	
4	

$$=\frac{\frac{21}{2}\times\frac{37}{4}}{\frac{5}{4}}$$

Or transferring denominators

21	5
4	2
37	4

$=\frac{21.4.37}{5.2.4}$ *pala*

---

1. आद्यन्तयोस्त्रैराशिकेऽभिन्नजाती प्रमाणमिच्छा च ।
फलमन्यजाति मध्ये तदन्त्यगुणमादिना विभजेत् ॥ —*Pāṭīgaṇita 43.*

2. चन्दनपलं सकर्षं सार्धैर्दशभिर्लभ्यते पणैर्यदि ।
तत्कियता लभ्यन्ते पलानि नव कर्षयुतानि ॥ —*Pāṭīgaṇita. Ex. 25.*

=4 purāna 13 panas, 2 kākinis and 16 varāṭakas (One purāna is equivalent to 16 panas one pana is equivalent to 4 kākinis and one kākini is equivalent to 20 varāṭakas or cowries

Inverse Rule of Three

This is known as *vyasta-trairāśika* (literally meaning inverse rule of three terms) After having described the rule of three Brahmagupta proceeds to give an account of this inverse rule of three

> Divide the *phala* with *iccha* and multiply by *pramāṇa* this gives the *vyasta trairāśika* inverse rule of three[1]

Here *pramāṇa* is the argument also known as the first term and, and *phala* is the fruit also known as the middle term and *iccha* is known as requisition or the last term As Bhāskara II clearly states this rule is applied where with the increase of the *iccha* the *phala* decreases or with its decrease the *phala* increases (*Līlāvatī*)

Rule of Compound Proportion

Brahmagupta and other writers call the rule of compound proportions as *pañca-rāśika* *sapta rāśika* etc meaning the rule of five terms rule of seven terms etc. depending on the number of terms involved in the problems There are sometimes grouped under the general application of the "Rule of Odd Terms" Āryabhaṭa I (499 A D) though actually gives the rule of three appears to have been quite familiar with the rule of compound proportion also In fact the difference between the rule of three and compound proportion is more or less artificial This view was expressed by Bhāskara I (525 A D) in his commentary on the *Āryabhaṭīya*

> Here Ācārya Āryabhaṭa has described the Rule of Three only How the well known Rules of Five etc are to be obtained ? I say thus The Ācārya has described only the fundamentals of *anupāta* (proportion) All others such as the Rule of Five etc. follow from that fundamental rule of proportion. How ? The Rule of Five etc. consist of combinations of the Rule of Three
>
> In the Rule of Five there are two Rules of

1 व्यस्तत्रैराशिक फलमिच्छा भक्त प्रमाण फलघात ।
[illegible] ॥ —*BrSpSi* XII 11

> Three, in the Rule of Seven three Rules of Three and so on This I shall point out in the examples

Brahmagupta gives the following rule relating to the solution of problems in compound proportion

> In the case of odd terms beginning with three terms up to eleven the result is obtained by transposing the fruits of both sides from one side to the other and then dividing the product of the larger set of terms by the product of the smaller s_t In all the fractions the transposition of denominators in like manner takes place on both sides[1]

This may be illustrated by taking an example from the commentary of Pṛthūdaka Svāmī on the *Brāhmasphuṭasiddhānta* ¬

Example —If there is an increase of 10 in 3 months on 100 (*niṣkas*) what would be the increase on 60 (*niṣkas*) in 5 months

Here the *Pramāṇa pakṣa* (the first set of terms) is
100 niskas 3 months 10 niṣkas (phala)

The second set or the *icchā pakṣa* is
60 *niṣkas* 5 months *x niṣkas*

The terms are written in compartments as below

100	60
3	5
10	0

In the above 10 (written lowest) is the *fruit* of the first side (*pramāṇa pakṣa*), and there is no *fruit* on the second side or the *iccā pakṣa* Interchanging the *fruits* we get

100	60
3	5
0	10

The larger set of terms is on the second side (*icchā pakṣa*) The product of the numbers is 3 000 The product of the

1 व्यस्त त्रैराशिक फलमिच्छा भक्त प्रमाणफलघात ।
त्रैराशिकादिषु फल विषमेष्वेकादशान्तेषु ॥
फलसक्रमणमुभयतो बहुराशि वधोऽल्पवधहृतो ज्ञेयम् ।
सकलेष्वेव भि नेषूभयतश्छेदसक्रमणम् ॥ —*BrSpSi* XII 11 12

number on the side of the smaller set of terms is 300 Therefore the required result is $\frac{3000}{300}=10$

Rule of Three as a Particular Case

According to Brahmagupta, the above method of compound proportion" may be applied to the Rule of Three. Taking the example solved under the Rule of Three

> If one *pala* and one *karṣa* of sandal wood are obtained for ten and a half *paṇas*, for how much will be obtained nine *palas* and one *karṣa* ?
>
> (4 karsas=1 pala)

We shall represent them according to the Rule of Compound Proportion as

Pramāṇa pakṣa 1 *pala* 1 *karṣa*, 10½ *paṇa*
or $\frac{5}{4}$ *pala* , $\frac{21}{2}$ *paṇa*

Icchā pakṣa 9 *pala*, 1 *karṣa* , x *paṇa*
or 37/4 *pāla* , x *paṇa*

This we shall represent as

5	37
4	4
21	0
2	

Transposing the fruits we have

5	37
4	4
0	21
	2

Transposing denominators

5	37
4	4
0	21
2	

The product of numbers on the side of the larger set is divided by the product of the numbers on the side of the smaller set 0 in this case is not a number It is the symbol for the unknown or absence. Hence the result is

$$\frac{37\cdot4\cdot21}{5\cdot4\cdot2} \text{ panas}$$

> The above method of working the Rule of Three is found among Arabs although it does not seem to have been used in India after Brahmagupta

Problem Containing Quadratic Equation

Perhaps Āryabhaṭa I is the first man in the history of mathematics to give a solution of a quadratic equation (499 A D) In his *Āryabhaṭīya, he gives a rule for the solution of the* following problem (I am reproducing it as described by Datta and Singh)

> The principal sum $p\,(=100)$ is lent for one month (interest unknown $=x$) This unknown interest is then lent out for $t(=\text{six})$ months After this period the *original interest* (x) plus *the interest on this interest* amounts to $A(=16)$ The rate interest (x) on the principal (p) is required

This problem requires the solution of the quadratic equation —

$$tx^2+px-AP=0$$

which gives $x=\dfrac{-p/2\pm\sqrt{(p/2)^2+Apt}}{t}$

The negative value of the radical does not give a solution of the problem, so that the result is

$$x=\frac{\sqrt{Apt+(p/2)^2}-p/2}{t}$$

This solution is stated by Āryabhaṭa I in the following words

> Multiply the sum of the interest on the principal and the interest (A) by the time (t) and by the principal (p) Add to this result the square of half the principal $(p/2)^2$ Take square-root this. Subtract half the principal $(p/2)$ and divide the remainder by the time (t) The result will be the (unknown) interest (x) on the principal [1]

Here the Sanskrit terms are *mūla* for principal and *phala* for interest

1 मूलफल सफल कालमूलगुणमर्धमूलकृतियुक्त ।
मूलं मूलार्धोनं कालहृत स्यात्स्वमूलफलम् ॥ *Ārya II 25*

Brahmagupta (628 A D) gives a more general rule He enunciates his problem thus

> The principal (p) is lent out for t_1 months and the unknown interest on this ($=x$) is lent out for t_2 months at the same rate and becomes A To find x

This evidently gives the quadratic

$$x^2+\frac{pt_1}{t_2}\,x-\frac{Apt_1}{t_2}=0$$

whose solution is

$$x=\pm\sqrt{\frac{Apt_1}{t_2}+\left(\frac{pt_1}{2t_2}\right)^2}-\frac{pt_1}{2t_2}$$

The negative value of the radical does not give a solution of the problem so it is discarded

Brahmagupta states the formula thus :

> Multiply the principal (p) by its time (t_1) and divide by the other time (t_2) (placing the result) at two places Multiply the first of these by the mixture (A) Add to this the square of half the other Take the square-root of this (sum) From the result subtract half the other This will be the interest (x) on the principal [1]

A Problem on Interest

Brahmagupta gives a solution of a problem on interest

> In what time will a given sum s the interest on which for t months is r become k times itself ?

The rule for the solution of this problem as given by Brahmagupta is

> The given sum multiplied by its time and divided by the interest (*phala*) being multiplied by the factor (*guna*) less one, is the time (*required*) [2]

Miscellaneous Problems

Brahmagupta in his *Ganitadhyaya* of the *Brahmasphuta-siddhanta* gives numerous solutions in relation to miscellaneous problems Here I shall be quoting a few of the problems which

---

1. [illegible] —*BrSpSi* XII 15
2. [illegible] —*BrSpSi* XII 14

have been quoted by his commentator Pṛthūdaka Svāmī in connection with one of his karaṇa-sūtra.[1]

> 1. A horse was purchased by (nine) dealers in partnership, whose contributions were one, etc. up to nine; and was sold by them for five less than five hundred. Tell me what was each man's share of the sale proceed[2]
> 2. Four colleges (maṭhas), containing an equal number of pupils, were invited to partake of a sacrificial feast. A fifth, a half, a third and a quarter (of the total number of pupils in the college) came from the respective colleges to the feast; and added to one, two, three and four, they were found to amount to eighty-seven; or, with those deducted, they were sixty seven. Find the actual number of the pupils that came from each college.[3]
>
> 3. Three (unequal) jars of liquid butter, of water and of honey, contained thirty-two, sixty and twenty-four *Pala* respectively; the whole was mixed together and the jars filled again. Tell me the quantity of butter, of water and of honey in each jar[4].

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1. प्रक्षेपयोगहृतया लब्ध्या प्रक्षेपका गुणा लाभाः ।
ऊनाधिकोद्धृतास्तद्युतोनया स्वफलमूलयुत ॥ *BrSpSi. XII. 16.*

2. एकाद्यैर्नवपर्यन्तैर्वणिग्भिर्मूलराशिभिः ।
क्रीतो हयोऽसौ विक्रीतः पञ्चोनैः पञ्चोनैः पंचभिः शतैः ।
किमेकैकस्य तत्रासीद् वृद्धि त्वं मित्रकान् मम ॥

3. मठस्थानानि चत्वारि छात्राणां समसंख्यया ।
भोक्तुं संमन्त्रितान्यासन् दीक्षायां किल यज्वना ॥
पंचार्धत्रिचतुर्थांशास्तेभ्यो भोक्तुं समागताः ।
एकद्वित्रिचतुर्युक्ता त्र्यशीतिः ससप्तका ॥
एवोत्तरैरथवा हीना सप्तषष्टिश्चतेऽशकाः ।
मठेभ्यश्छात्रसंख्या मे ब्रूहि ये चागता यतः

4. घृतोदक मधुना ये त्रयः कलसकाः पलैः ।
रदषष्टिजिनैः पूर्णा एकीभूतास्ततः पुनः ॥
मिश्रेण पूरिता यावत् तावत् संख्या न वेद्म्यहम् ।
घृतोदकमधुना तामेकैकत्र गता वद ॥

—: o :—

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CHAPTER IX

Brahmagupta as an Algebraist

Ancient Indian name for algebra is *Bijaganita* where *bija* means *element* or *analysis* and *ganita* stands for the *science of calculation* As early as 860 A.D, Pṛthūdaka Svāmi used this epithet for algebra in his commentary Brahmagupta calls algebra as *Kuṭṭakaganita or merely kuṭṭaka, a term which was* later on used for "*pulveriser* which deals with that special section of algebra which is connected with indeterminate equations of the first degree Algebra is often also known as *avyakta ganita* or the calculations with *unknowns*, in contrast to arithmetic which was known as *vyakta ganita* or the calculations with *knowns*

Algebra goes to Europe from India

In the history of mathematical sciences, as Colebrooke rightly remarks, it has long been a question to whom the invention of algebraic analysis is due. There is no doubt that Europe got algebra from Arabs mediately or immediately But the Arabs themselves scarcely pretend to the discovery of algebra Colebrooke says that they were not in general inventors but scholars during the short period of their successful culture of the sciences, and the germ at least of the algebraic analysis is to be found among the Greeks in an age not precisely determined but more than probably anterior to the earliest dawn of civilisation among the Arabs, and this science in a more advanced state subsisted among the Hindus prior to the earliest disclosure of it by the Arabians to modern Europe. (Colebrooke *Dissertation on the Algebra of the Hindus*)[1]

Colebrooke based his observations on the texts he could procure for his studies. These were: Bhāskara II's *Bijaganita* or *Vijaganita* (1150A D) and *Lilavati* (1150 A D) the *Ganitādhyāya* and *Kuṭṭakādhyāya* of Brahmagupta in his famous treatise the *Brahma Siddhānta* or rather the *Brahmasphuṭasiddhānta* (628

1. Colebrooke H. T *Miscellaneous Essays* Vol II, 1872 p 413.

A.D.). There can be no doubt regarding the age of these two authors. Bhāskara II completed his great work on the *Siddhānta-śiromaṇi* in 1072 Śaka, and *Karaṇa-kutūhala* a practical astronomical treatise in 1105 Śaka; these dates are based on the passages given by Bhāskara himself in his works. The *Bija-gaṇita* and the *Līlāvatī* form parts of the great treatise, the *Siddhānta śiromaṇi*. The genuineness of the text is established, as Colebrooke says, with no less certainty by numerous commentators in Sanskrit, besides a Persian version of it. Those commentaries comprise a perpetual gloss, in which every passage of the original is noticed and interpreted : and every word of it is repeated and explained. From comparison and collation of various texts, it appears then that the work of Bhāskara, exhibiting the same uniform text which the modern transcripts of it do, was in the hands of both Muhammedans and Hindus, between two or three centuries ago : and numerous copies of it having been diffused throughout India, at an earlier period, as of a performance held in high estimation. It was the subject of study and habitual reference in countries and places so remote from each other as the north and west of India and the Southern Peninsula.

This though not marking any extraordinary antiquity, nor approaching to that of the author himself, was a material point to be determined : as there will be in the sequel, so says Colebrooke, occasion to show, that modes of analysis, and in particular, general methods for the solution of indeterminate problems both of the first and second degrees, are taught in the *Bija gaṇita*, and those for the first degrees repeated in the *Līlāvatī*, which were unknown to the mathematicians of the West, until invented anew in the last two centuries by algebraists of France and England.[1] Bhāskara who himself flourished more than six hundred and fifty years ago, was in this respect a compiler and took those methods from Indian authors as much more ancient than himself.

Regarding the age of the precursors of Bhāskara II, Colebrooke says : The age of his precursors cannot be determined with equal precision. He then proceeds to examine the evidence as follows :

1. Colebrooke, H. T., *Miscellaneous Essays*, p. 421.

Bhāskara and by other astronomical writers, and the title of the work, *Brāhmasiddhānta* or sometimes *Brāhmasphuṭasiddhānta*, corresponds, in the shorter form, to the known title of Brahmagupta's treatise in the usual references to it by Bhāskara's commentators, and answers, in the longer form, to the designation of it, as indicated in an introductory couplet which is quoted from Brahmagupta by Lakṣmīdāsa, a scholiast of Bhāskara II. Remarking this coincidence, the translator proceeded to collate with the text and commentary, numerous quotations from both, which he found in Bhāskara's writings or in those of his expositors The result confirmed the indication and established the identity of both text and scholia as Brahmagupta's treatise, and the gloss of Pṛthūdaka The authenticity of the *Brāhmasiddhānta* is further confirmed by numerous quotations in the commentary of Bhaṭṭotpala on the *Samhitā* of Varāhamihira as the quotations from the *Brāhmasiddhānta* in that commentary, (which is the work of an author who flourished eight hundred and fifty years ago) are verified in the copy under consideration A few instances of both will suffice, and cannot fail to produce conviction

It is confidently concluded that the chapters on arithmetic and algebra fortunately entire in a copy in many parts imperfect, of Brahmagupta's celebrated work as here described, are genuine and authentic It remains to investigate the age of the author

Mr Davis, who first opined to the public a correct view of the astronomical computations of the Hindus, is of opinion, that Brahmagupta lived in the seventh century of the Christian era Dr William Hunter, who resided for some time with a British Embassy at Ujjayini and made diligent researches into the remains of Indian science at that ancient seat of Hindu astronomical knowledge was there furnished by the learned astronomers whom he consulted with the ages of the principal ancient authorities They assigned to Brahmagupta the date of 550 Saka, which answers to A D

628 The grounds on which they proceeded are unfortunately not specified: but as they gave Bhāskara's age correctly, as well as several other dates right, which admit of being verified, it is presumed that they had grounds, *though unexplained, for the information which they* communicated

Mr Bentley who is little disposed to favour *the antiquity of an Indian astronomer, has given his reasons for considering* the astronomical system which *Brahmagupta* teaches, to be between twelve and thirteen hundred years old (1263 years in A D 1799) Now as the system taught by this author is professedly one corrected and adapted by him to conform with the observed positions of the celestial objects when he *wrote the age, when their positions would be conform*able with the *results of computations made as by him* directed, is precisely the age of the author himself and so far as Mr Bentley s calculations may be considered to approximate the truth the date of Brahmagupta s performance is determined with like approach to exactness, within a certain latitude however of uncertainty for allowance to be made on account of the inaccuracy of Hindu observations

The translator has assigned on former occasions the grounds upon which he sees reason to place the author's age, soon after the period when the vernal equinox *coincided* with the beginning of the lunar mansion and zodiacal *asterism Aśvinī* where the *Hindu ecliptic* now commences. *He is supported in it by the sentiments* of *Bhāskara and other Indian astronomers,* who infer from *Brahmagupta's* doctrine concerning the solstitial points of which he does not admit a periodical motion that he lived when the equinoxes did not sensibly to him deviate from the beginning of *Aśvinī* and middle of *citrā* on the Hindu sphere. On these grounds it is maintained that Brahmagupta is *rightly* placed in the sixth or beginning of the seventh *century* of the Christian era, as the subjoined *calculations* will more particularly show The age when *Brahmagupta*

flourished, seems then from the concurrence of all these arguments, to be satisfactorily settled as antecedent to the earliest dawn of the culture of sciences among the Arabs, and *consequently establishes* the fact that *the Hindus were in possession of algebra before* it was known to the Arabians

Brahmagupta's treatise, however, is not the earliest work known to have been written on the same subject by an Indian author The most eminent scholiast of Bhāskara II (Gaṇeśa) quotes a passage of Āryabhaṭa specifying algebra under the designation of *Bija*, and making separate mention of *Kuṭṭaka*, which more particularly intends a problem subservient to the general method of resolution of indeterminate problems of the first degree . he is understood by another of Bhāskara's commentators to be at the head of the elder writers to whom the text then under consideration adverts, as having designated by the name of *Madhyamāharaṇa* the resolution of affected quadratic equations by means of the completion of the square It is to be presumed, therefore, that the treatise of Āryabhata then extant, did extend to quadratic equations in the determinate analysis, and to indeterminate problems of the first degree, if not to those of the second likewise, as most probably it did

This ancient astronomer and algebraist so says Colebrooke, was anterior to both Varāhamihira and Brahmagupta, being repeatedly named by the latter, and the determination of the age when he flourished is particularly interesting as his astronomical system, though on some points agreeing, essentially disagreed on others, with that which the Hindu astronomers still maintain

He as Colebrooke says is considered by the commentators of the *Sūryasiddhānta* and *Śiromaṇi* as the earliest of uninspired and mere human writers on the science of astronomy, as having introduced requisite corrections into the system of Parāśara from whom he took the numbers for the planetary mean motions, as

having been followed in the tract of emendation after a sufficient interval to make further correction requisite, by Durgāsinha and Mihira, who were again succeeded after a further interval by Brahmagupta, son of Jiṣnu

In short, says Colebrooke Āryabhaṭa was founder of one of the sets of Indian astronomers, as Puliśa an author likewise anterior to both Varāhamihira and Brahmagupta, was of another: which were distinguished by *names derived from the discriminative tenets* respecting the commencement of planetary motions at sunrise according to the first, but at midnight according to the latter, on the meridian of Laṅkā, at the beginning of the great astronomical cycle A third sect began the astronomical day as well as the great period at noon

Āryabhaṭa's name accompanied the intimation which the Arab astronomers (under the Abbasside Khalifs, as it would appear,) received, that three distinct astronomical systems were current among the Hindus of those days and it is but slightly corrupted, certainly not at all disguised, in the Arabic representation of it *Arjabahar*, or rather *Arjabhar*, (corrupted form of Āryabhaṭa) The two other systems were first Brahmagupta's *Siddhanta* which was the one they became best acquainted with and to which they apply the denomination of the *sind hind*, and second, that of *Arca*, the Sun, which they write *Arcand* a corruption still prevalent in the vulgar Hindi.

Āryabhaṭa appears to have had more correct notions of the true explanation of celestial phenomena than Brahmagupta himself so says Colebrooke, who in a few instances correcting errors of his predecessor, but oftener deviating from that predecessor s juster views has been followed by the herd of modern Hindu astronomers, in a system not improved, but deteriorated since the time of the more ancient author

Considering the proficiency of Āryabhaṭa in astronomical science, and adverting to the fact of his having

written algebra, as well as to the circumstance of his being named by numerous writers as the founder of a sect, or author of a system in astronomy, and being quoted at the head of algebraists, when the commentators of extant treatises have occasion to mention early and original writers on this branch of science, it is not necessary to seek further for a mathematician qualified to have been the great improver of the analytic art, and likely to have been the person by whom it was carried to the pitch to which it is found to have attained among the Hindus, and at which it is observed to be nearly stationary through the long lapse of ages which have since passed : the later additions being few and unessential in the writings of Brahmagupta, of Bhāskara and of Jñānarāja, though they lived at intervals of centuries from each other.

Āryabhaṭa, Colebrooke rightly says, then being the earliest author known to have treated of Algebra among the Hindus, and being likely to be, if not the inventor, the improver of that analysis, by whom too it was pushed nearly to the whole degree of excellence which it is found to have attained among them; it becomes in an especial manner interesting to investigate any discoverable trace in the absence of better and more direct evidence, which may tend to fix the date of his labours; or to indicate the time which elapsed between him and Brahmagupta, whose age is more accurately determined.

Taking Āryabhaṭa, for reasons given, to have preceded Brahmagupta and Varāhamihira by several centuries; and Brahmagupta to have flourished more than twelve hundred years ago, and Varāhamihira, concerning whose works and age, Colebrooke has given a few notes, and has placed him at the beginning of the sixth century after Christ, it appears probable that this earliest of known Hindu algebraists wrote as far back as the fifth century of the Christian era; and perhaps in an earlier age. Hence it is concluded that he is nearly as ancient as the Grecian algebraist Diophantus, sup-

posed on the authority of Abulfaraj to have flourished in the time of the Emperor Julian or about A D 360

Colebrooke further says Admitting the Hindu and Alexandrian authors to be nearly equally ancient it must be conceded in favour of the Indian algebraist that he was more advanced in the science since he *appears to have been in possession of the resolution* of equations involving several unknowns, which it is not clear nor fairly presumable that Diophantus knew and a general method of indeterminate problems of at least the first degree to a knowledge of which the Grecian algebraist had certainly not attained though he displays infinite sagacity and ingenuity in particu*lar solutions and though a certain routine is indiscer*nible in them

Colebrooke appears to be of the view that Greeks were the first to discover the solution of equations involving one unknown and this knowledge was passed to ancient Indians by their Greek instructors in improved astronomy But by the ingenuity of the Hindu scholars the hint was rendered *fruitful and the algeb*raic method was soon ripened from that slender beginning to the advanced state of a well arranged science as it was taught by Āryabhaṭa and as it is found in treatises compiled by Brahmagupta and Bhāskara

We do not agree with this analysis in entirety Indian algebra is entirely of Indian roots It had its beginning in the times of Saṃhitās and Brāhmaṇas Some of the equations and problems were solved by geometric methods *It must have had its origin in the Śulba* period if not before Āryabhaṭa undoubtedly was the discoverer of many algebraic solutions of equations of the first and higher order with one and more unknowns It is rather too much to trace the influence of Diophantus on Indian algebra which developed in this country independently Brahmagupta is one of the most brilliant algebraists we ever had in the entire history of mathematics.

Technical Terms

Coefficient—

In the ancient Indian algebra, there is no systematic term for the coefficient. Usually, the power of the unknown is mentioned when the reference is to the coefficient of that power. At one place, for example, we find Pṛthūdaka Svāmī (the commentator of Brahmagupta's *Brāhmasphuṭasiddhānta*) writing "the number (*aṅka*) which is the coefficient of the square of the unknown is called the '*square*' and the number which forms the coefficient of the (simple) unknown is called the 'unknown quantity' (*avyakta-māna*)."[1] However, at many places, we find the use of a technical term also. Brahmagupta once calls the coefficient *saṁkhyā*[2] (number) and on several other occasions *gunaka*[3] or *guṇakāra*[4] (multiplier). Pṛthūdaka Svāmī (860 A.D.) calls it *aṅka* (number) or *prakṛti* (multiplier) These terms may also be seen in the works of *Śrīpati*[6] (1039) and Bhāskara II[7] (1150 A.D.). The former also used the word *rūpa* for the same purpose.[8]

Unknown Quantity

The unknown quantity has been termed as *yāvat-tāvat* (meaning so-much-as or as-many-as) in literature as early as 300 B. C. (vide the *Sthānāṅga-sūtra*[9]). In the *Bakhaśālī Manuscript*, it has been termed as *yadṛcchā*, *vāñchā* or *kāmikā* (or any desired quantity)[10]. Āryabhaṭa I in one of his verses calls the unknown as *gulikā*[11] (literally meaning a shot) From the early seventh century A.D., the word *avyakta* was used for unknown quantities. Brahmagupta uses this term in his *Brāhmasphuṭasiddhānta*[12]

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1. *BrSpSi. XVIII. 44* (Com)
2. वर्णप्रमाणं भावितघातो भवतीष्ट वर्णं संख्यैवम् । —*BrSpSi. XVIII 63*
3. मूलं द्विधेष्ट वर्गाद् गुणक गुणादिष्ट युत विहीनाच्च । —*BrSpSi. XVIII 64*
   वर्गच्छिन्ने गुणके प्रथमं तन्मूलं भाजितं भवति । —*BrSpSi. XVIII 70*
4. प्रथमान्त्यमूलमन्यो गुणकार पदोद्धृतः प्रथमः । —*BrSpSi. XVIII 69*
5. *BrSpSi XVIII. 44* (Com)
6. *Siśe XIV 33-5.*
7. *Bījagaṇita*
8. *Siśe XIV 33-5.*
9. *Sūtra 747.*
10. *BMs.* Folio 22, verso, 23, recto and verso.
11. गुलिकान्तरेण विभजेद् द्वयोः ।
12. अव्यक्तवर्ग घनवर्ग वर्गपञ्चगत षड्गतादीनाम् ।

portion of the equation whilst its other part is practically invisible or unknown[1]

Unknowns and Symbolism

Āryabhaṭa I (499 A D) probably used coloured *gulikās* or shots for represtnting different unknowns Brahmagupta mentions *varṇa* as the symobls for unknown He has however not indicated how these *varṇas* or colours were used as symbols for unknowns Perhaps we might conculude from this that the method of using colours as symbols for unknown quantities was very common and familiar to the algehraists Datta and Singh say that the Sanskrit word *varṇa* denotes colour as well as a letter of alphabet and therefore, letters of alphabet came into use for unknown quantities *kālaka* (black) *nīlaka* (blue) *pītaka* (yellow) *lohita* (red) *haritaka* (green) *śvetaka* (white) *citraka* (variegated) *kapilaka* (tawny) *piṅgalaka* (reddish brown) *dhūmraka* (smoke coloured) *pāṭalaka* (pink) *śavalaka* (spotted) *śyāmalaka* (blackish) *mecaka* (dark blue) etc[2]

It should be further noted that the first unknown quantity *yāvat tāvat* is not a *varṇa* or colour It thus clearly indicates that the use of colours as symbols came at a later stage whilst the word *yāvat tāvat* was in currency from much earlier times Some authorities think that the term *yāvat tāvat* is a corrupted form of *yāvakastāvat* (where *yāvaka* means red) Pṛthūdaka Svāmi has sometimes used the term *yavaka* for an unknown quantity[3]

Laws of Signs

Brahmagupta has in his Chaper XVIII devoted a special section entitled *Dhanarṇa Śunyānām Samkalanam* or calculations dealing with quantities bearing positive and negative signs and zero

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1 अव्यक्तान्तर भक्त व्यस्त रूपान्तर समेऽव्यक्त । —*BrSpSi* XVIII 43
वर्ग चतुर्गुणिताना रूपाणा मध्यवर्ग सहितानाम् ॥ —*BrSpSi* XVIII 44

2 यावत् तावत् कालको नीलकोऽन्यो वर्ण पीतो लोहितश्चैतदाद्या ।
अव्यक्ताना कल्पिता मानसज्ञास्तत्सख्यान कर्तु माचार्यवर्यै ॥
यावत् तावत् कालक नीलक पीताश्च लोहितो हरित ।
श्वेतक चित्रक कपिलक पाटलका पाण्डु धूम्र शबलाश्च ॥
श्यामलक-मेचक-धवलक पिशङ्ग-सारङ्ग-बभ्रु गौराद्या । —Nārāyana *Bījagaṇita*

3 *BrSpSi XII 15* (Com) XII 18

Regarding *addition* Brahmagupta says

> The sum of two positive numbers is positive of two negative numbers is negative of a positive and negative number is the difference[1]

Regarding *subtraction* Brahmagupta further says

> From the greater should be subtracted the smaller (the final result is) positive if positive from positive and negative if negative from negative If however the greater is subtracted from the less that difference is reversed (in sign) negative becomes positive and positive becomes negative When positive is to be subtracted from negative or negative from positive then they must be added together[2]

Mahāvira (850 A D) Bhāskara II (1150 A D) and Nārāyana (1350 A D) have also given similar rules regarding addition (*Saṅkalanam*) and the subtraction (*vyavakalanam*)

Again the rule given by Brahmagupta regarding *Multiplication* is as follows

> The product of a positive and a negative (number) is negative of two negatives is positive positive multiplied by positive is positive[3]

His rule regarding *division* is as follows

> Positive divided by positive or negative divided by negative becomes positive But positive divided by negative and negative divided by positive remains negative[4]

Similar rules for multiplication and division were provided by later authorities as Mahāvira and Bhāskara II

1 *BrSpSi XII 15 (Com) XII 18 (Com)*

2 धनयोर्धनमृणमृणयोर्धनर्णयोरन्तरं समैक्यं खम् ।
ऋणमैक्यं च धनमृणधनशून्ययोः शून्ययोः शून्यम् ॥ —*BrSpSi* XVIII 30

3 ऊनमधिकाद्विशोध्यं धनं धनादृणमृणादधिकमूनात् ।
व्यस्तं तदन्तरं स्यादृणं धनं धनमृणं भवति ॥
शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम् ।
शोध्यं यदा धनमृणादृणं धनाद्वा तदा क्षेप्यम् ॥ —*BrSpSi* XVIII 31 32

4 ऋणमृणधनयोर्घातो धनमृणयोर्धनवधो धनं भवति ।
शून्यर्णयोः खधनयोः खशून्ययोर्वा वधः शून्यम् ॥ —*BrSpSi* XVIII 33

5 धनभक्तं धनमृणहृतमृणं धनं भवति खं खभक्तं खम् ।
भक्तमृणेन धनमृणं धनेन हृतमृणमृणं भवति ॥ —*BrSpSi* XVIII 34

Brahmagupta lays down the rules regarding *evolution* and *involution* as follows

> The square of a positive or a negative number is positive . The (sign of the) root is the same as was that from which the square was derived[1]

As regards the latter portion of this rule Pṛthūdaka Svāmi has the following comment to make ' The square root should be taken either negative or positive as will be most suitable for subsequent operations to be carried on

It will be interesting to observe the following observation of Mahāvira (850 A D) regarding square-root of a negative quantity Since a negative number by its own nature is not a square it has no square root [2] So says Śripati A negative number by itself is non-square so its square-root is unreal so the rule (for the square-root) should be applied in the case of a positive number "[3]

Algebraic Operations

Brahmagupta and other algebraists recognise six operations as fundamental in algebra addition, subtraction multiplication division, squaring and the extraction of the square-root

Regarding *addition* and *subtraction* Brahmagupta says

> Of the unknowns their squares, cubes, fourth powers, fifth powers, sixth powers etc addition and subtraction are (performed) of the like of the unlike (they mean *simply their) statement severally*[4]

. In place of of the like Bhāskara II uses the term of those of the same species (*jāti*) amongst unknowns

> Addition and subtraction are performed of those of the same species (*jāti*) amongst unknowns of different species they mean their separate statement [5]

1 खोद्धृत मृण धन वा तच्छेद खमृणधनविभक्त वा ।
ऋणधनयोर्वर्ग स्वे खे खस्य पद कृतिर्यत् तत् ॥ — *BrSpSi* XVIII 35

2 *GSS* I 52

3 Ś *ie* XIV 5

4 अव्यक्तवर्ग घनवर्ग वर्ग पञ्चगत षड्गतादीनाम् ।
तुल्याना सकलित व्यवकलिते पृथगतुल्यानाम् ॥ —*BrSpSi XVIII 41*

5 योगोऽन्तरं तेषु समान जात्योर्विभिन्न जात्योश्च पृथक स्थितिश्च ।
— *Bhāskara* II *Bijagaṇita*

This means that the numerical coefficients of x cannot be added to or subtracted from the numerical coefficients of y or x^2 or x^3 or xy and so on because these terms belong to different *jati* or they do not belong to the category of the *like*

Again regarding *multiplication* Brahmagupta says

> The product of the two like unknowns is a square the product of three or more like unknowns is a power of that designation The multiplication of unknowns of unlike species is the same as the mutual product of symbols it is called *bhavita* (product or factum)[1]

Having given the rules of the operations for addition subtraction and multiplication Brahmagupta does not think it necessary to deal with other operations His section on the calculations with zero, negative and positive quantities ends here

How is an Equation Formed?

Pṛthūdaka Svāmi while commenting on a verse in *Brahmasphuṭasiddhānta* speaks as follows

> In this case in the problem proposed by the questioner *yāvat tāvat* is put for the value of the unknown quantity Then performing multiplication division etc as required in the problem the two sides shall be carefully made equal The equation being formed in this way then the rule (for its solution) follows[2]

Plan for Writing Equations

When in regards to a given problem an equation has been formed it has to be written down for further operations This writing down of an equation is technically known as *nyāsa* Perhaps the oldest record of *nyāsa* is to be found in the *Bakhshālī Manuscript* According to the procedure prescribed in this work the two sides of an equation are put down one after the other in the same line without any sign of equality being interposed Thus the equations

$$\sqrt{x+5}=s \quad \sqrt{x-7}=t$$

appear as

---

1 सदृशद्विवधो वर्गस्त्र्यादिवधस्तद्गतोऽन्यजातिवधः ।
अन्योन्यवर्णघातो भावितकः पूर्ववच्छेषम् ॥ —*BrSpSi* XVIII 42.

2. *BrSpSi* XVIII 43 (com)

0 5 *yu* *mū* 0	*sa* 0 7 + *mū* 0
1 1 1	1 1 1

Here *yu* (यु) stands for *yuta* (युत), meaning added, subtraction is+sign, derived from Ksaya or (क्षय) meaning diminished, *gu* (गु) for *guna or gunita*, meaning multiplied; *bhā* (भा) for division from *bhājita* and *mū* (मू) for square-root, from *mūla* meaning root; *zero* (∘) was used to mark a vacant place.

Again, the following equation

$$x+2x+3\times3x+12\times4x=300$$

is represented as ;

0	2 1	3 3	12 4	*dṛśya* 300
1	1 1	1 1	1 1	

There is no sign for unknown in the *Bakhaśālī Manuscript*.

Later on this plan of writing equations as adopted in *Bakhaśālī Manuscript* was abandoned in India; a new one was adopted in which the two sides are written one below the other without any sign of equality. It must be stated that in this new plan the term of similar denominations are usually written one below the other and even the terms of absent denominations on either side are clearly indicated by putting zeros as their coefficients. We find a reference to this new plan in the algebra of Brahmagupta.

> From which the square of the unknown and the unknown are cleared, the known quantities are cleared (from the side) *below* that[1].

Here in this verse, the words "*adhastāt*" clearly indicate *that one side of the equation is written below the other.*, *As* an illustration, Prthūdaka Svāmī represented the equation[2] :—

$$10x-8=x^2+1$$

as :

yā va 0 *yā* 10 *rū* 8̇ $(x^2.0+x.10-8)$

yā va 1 *yā* 0 *rū* 1 $(x^2.1+x.0+1)$

which means, x^2 was written as *yāvat-varga* (*yā va*) and x was written as *yāvat* or *yā*. The minus sign was represented by a dot at the top of the number.(—8 was written as 8̇). We shall take another illustration from Pṛthūdaka Svāmī

He would write the equation

$$197x-1644y-z=6302$$

as

1. *BrSpSi*. XVII. 43, compare also Bhāskara II, *Bijaganita*.
2. *BrSpSi*. XVIII. 49 (*com*)

yā 197 *kā* 1644 *nī* 1 *rū* 0
yā 0 *kā* 0 *nī* 0 *rū* 6302

Here the first unknown x is represented by *yā(vat)* the second unknown y by *kā(laka)* and the third unknown z by *nī (laka)* and the term without unknown a mere number is written by *rū(paka)* The two sides one written below the other if written in the present form would appear as

$197x - 1644y - z + 0 = 0x + 0y + 0z + 6302$

The *Bijagaṇita* of Bhāskara II also follows the same procedure One instance from it would be quoted here to illustrate the method of expressing equations

$8x^3 + 4x^2 + 10y^2x = 4x^3 + 12y^2x$

or $8x^3 + 4x^2 + 10y^2x = 4x^3 + 0x^2 + 12y^2x$

is written as follows on Bhāskara s or Brahmagupta s plan

x^3 is *ghana* of *yāvat* (abbreviated as *yā gha*)
x^2 is *varga* of *yāvat* (abbreviated as *yā va*)
y^2 is *varga* of *kālaka* (abbreviated as *kā va*)
the coefficients 10 and 12 are *bhāvita* (abbreviated as *bhā*)

The equation is

yā gha 8 *yā va* 4 *kā va yā bhā* 10
yā gha 4 *yā va* 0 *kā va yā bhā* 12

Datta and Singh state that the use of the old plan of writing equations is sometimes met with in later works also For instance in the MS of Pṛthūdaka Svāmi s commentary[1] on the *Brāhmasphuṭasiddhānta* an incomplete copy of which is preserved in the library of the Asiatic Society of Bengal (No I B6) we find a statement of equations thus 'first side *yāvargaḥ 1 yāvakaḥ* 200 *rū* 0 second side *yāvargaḥ* 0 *yāvakaḥ* 0 *rū* 1500

Sodhana or Clearance of an Equation

After *nyāsa* or statement of an equation the operation to be performed is known as *śodhana* (clearance) or *samśodhana* (equi-clearance or complete clearance) The nature of this clearance varies according to the kind of equation In the case of an equation in one unknown only whether linear

1 *BrSpSi* XII 15 (*com*)

quadratic or of higher powers, one side of it is cleared of the unknowns of all denominations and the other side of it of the absolute terms, so that the equation is ultimately reduced to one of the form

$$ax^2+bx=c,$$

where a, b, c may be positive or negative; some of them may even be zero. Thus Brahmagupta observes:

> From which the square of the unknown and the unknown are cleared, the known quantities (*rūpāṇi*) are cleared (from the side) *below* that.[1]

On this Pṛthūdaka Svāmī comments as follows:

> This rule has been introduced for that case in which the two sides of the equation having been formed in accordance with the statement of the problem, there are present the square and other powers of the unknown together with the (simple) unknown. The absolute terms should be cleared off from the side opposite to that from which are cleared the square (and other powers) of the unknown and the (simple) unknown. When perfect clearance (*saṁśodhana*) has been thus made...[2]

Śrīdhara and Bhāskara II have also given the rules of clearance almost on the same lines. Thus the equation

yā va 0 *yā* 10 *rū* $\dot{8}$
yā va 1 *yā* 0 *rū* 1

after perfect clearance having been made will be (according to Pṛthūdaka Svāmī)

yā va 1 *yā* 10 *rū* $\dot{9}$

i.e. the equation $10x-8=x^2+1$
after clearance would become

$$x^2-10x=-9.$$

Classification of Equations

Usually equations are classified as:

simple equation *yāvat tāvat*
quadratic *varga*

---

1 [illegible] — *BrSpSi*. XVIII. 43.
2. *BrSpSi*. XVIII. 43 (*com.*)

cubic · *ghana*
biquadratic *varga-varga*

Brahmagupta classified them as

(i) equations in one unknown quantity. *eka-varṇa samikaraṇa*

(ii) equations in several unknowns *aneka-varṇa samikaraṇa*

(iii) equations involving products of unknowns. *bhāvita*

Eka-varṇa samikaraṇas (equations with one unknown) are further divided into (i) linear equations, and (ii) quadratic equations (*avyakta varga samikarana*)

Pṛthūdaka Svāmi has classified equations in a different manner as follows

(i) linear equations with one unknown: *eka-varṇa samikarana*

(ii) linear equations with more unknowns *aneka-varṇa samikaraṇa*

(iii) equations with one two or more unknowns in their second or higher powers *madhyamāharana*

(iv) equations involving products of unknowns *bhāvita*

As the method of solution of an equation of the third class (i e equations with one or several unknowns in their second or higher powers) is based upon the principle of the elimination of the middle term, that class is called by the term *madhyama* (middle) *āharaṇa* (elimination) The classification of Brahmagupta and Pṛthūdaka Svāmi more or less received recognition by later writers on algebra as Bhāskara II and others

Linear Equations with One Unknown and Their Solutions

The first solution of a linear equation with one unknown is obtainable in the *Śulba Sūtras* but not through an algebraic process,—the Śulba process is geometrical It is said that there is a reference in the *Sthānāṅga Sūtra* (c 300 B C) to a linear equation by its name *yāvat tāvat* There has been a good deal of

of the coefficients on the unknown (*gulika*) The quotient will be the value of the unknown (*gulika*), if their possessions be equal [1]

The original verse contains the term "*gulikantara*" which has been here translated as the difference of the coefficients of the unknowns We have already stated earlier that Āryabhaṭa uses the term *gulika* or shot for an unknown quantity (*gulikāntara* literally means only the difference of unknowns) This practice is also followed by other Indian algebraists Pṛthūdaka Svāmī rightly observed that according to the usual practice in this country, 'the coefficient of the square of the unknown is called the square (of the unknown) and the coefficient of the (simple) unknown is called the unknown [2]

The rule given by Āryabhaṭa then contemplates a problem of this kind

> Two persons who are equally rich, possess respectively a, b times a certain unknown amount together with c, d units of money in cash What is that amount ?

Now if x be the amount unknown, then according to the problem

$$ax+c=bx+d$$

Thence

$$x=\frac{d-c}{a-b}$$

Āryabhaṭa has merely expressed this solution in his language

Regarding the solution of linear equations Brahmagupta says

> In a (linear) equation in one unknown the difference of the known terms taken in the reverse order divided by the difference of the coefficients of the unknown (is the value of the unknown) [3]

---

1 गुलिकान्तरेण विभजेद् द्वयो पुरुषयोस्तु रूपकविशेषम् । लब्धं गुलिकामूल्यं यद्यर्थकृतं भवति तुल्यम् ॥ —*Arya* II 30

2 *BrSpSi*, XVIII 44 (*com*)

3 अव्यक्तान्तरभक्तं व्यस्तं रूपान्तरं समेऽव्यक्तः । वर्गाव्यक्ताः शोध्या यस्माद् रूपाणि तदधस्तात् ॥ —*BrSpSi* XVIII 43

Similar solutions have been offered by the other Indian algebraists who followed Brahmagupta like Śrīpati, Bhāskara II and Nārāyana. Here again, we take a problem proposed by Brahmagupta in this connection :

Problem :

> Tell the number of elapsed days for the time when four times the twelfth part of the residual degrees increased by one, plus eight will be equal to the residual degrees plus one.[1]

Pṛthūdaka Svāmī has solved this problem as follows :

Here the residual degrees are (put as) *yāvat-tāvat*, *yā*; increased by one, *yā* 1 *rū* 1; twelfth part of it, $\frac{yā\ 1\ rū\ 1}{12}$; four times this, $\frac{yā\ 1\ \ rū\ 1}{3}$; plus the absolute quantity eight, $\frac{yā\ 1\ \ rū\ 25}{3}$. This is equal to the residual degrees plus unity. The statement of both sides tripled is

yā 1 *rū* 25
yā 3 *rū* 3

This difference between the coefficients of the unknown is 2 By this the difference of the absolute terms, namely 22, being divided, is produced the residual of the degrees of the Sun, 11. These residual degrees should be known to be irreducible. The elapsed days can be deduced then, (proceeding) as before.

If put in the modern notations, it means the solution of the equation :

$$\frac{4}{12}(x+1)+8=x+1,$$

from which we have

$$x+25=3x+3$$

or $$2x=22$$

or $$x=11.$$

Rule of Concurrence or Saṃkramaṇa

Brahmagupta has included this rule in algebra, whereas other Indian mathematicians included it in arithmetic. *Saṁ-*

1. सैकांशकशेषाद् द्वादशभागश्चतुर्गुणोऽष्टयुतः ।
सैकांशशेषतुल्यो यदा तदाऽहर्गणं कथय ॥ —*BrSpSi. XVIII. 46*

kramaṇa is the solution of the simultaneous equations of the type

$$x+y=a$$
$$x-y=b$$

Brahmagupta's rule for solution is

> The sum is increased and diminished by the difference and divided by two (the result will be the two unknown quantities) (this is) concurrence (*Saṃkramaṇa*)[1]

Brahmagupta restates this rule in the form of a problem and its solution

> The sum and difference of the residues of two (heavenly bodies) are known in degrees and minutes What are the residues? The difference is both added to and subtracted from the sum and halved (the results are) the residues[2]

Linear Equations with Several Unknowns

The first mention of a solution of the problem with more than one unknown is found in the *Bakhasālī Manuscript* and a system of linear equations of this type is solved in the Bakhaśālī treatise substantially by the False Position Rule

A generalised system of linear equations will be

$$b_1\Sigma x-c_1x_1=a_1 \quad b_2\Sigma x-c_2x_2=a_2$$
$$b_n\Sigma x-c_nx=a$$

Therefore

$$\Sigma x=\frac{\Sigma(a/c)}{\Sigma(b/c)-1}$$

Hence

$$x_r=\frac{b_r}{c_r}\times\frac{\Sigma(a/c)}{\Sigma(b/c)-1}-\frac{a_r}{c_r}$$
$$r=1\ 2\ 3 \quad n$$

One particular case, where $b_1=b_2=b_3= \quad =b_n=1$ and $c_1=c_2=c_3= \quad =c_n=c$ has been treated by Brahmagupta at one place He gives the rule as follows

---

1 योगोऽन्तरयुतहीनो द्विहृत संक्रमणमन्तरविभक्त वा ।
वर्गान्तरमन्तरयुतहीन द्विहृत विषमकर्म ॥ —*BrSpSi XVIII 36*

2 भागकला विवरैक्यं दृष्ट्वा विकलान्तरं च के शेषे ।
ऐक्यं द्विधाऽन्तराधिक हीनं च द्विभाजित शेषे ॥ —*BrSpSi XVIII 96*

> The total value (of the unknown quantities) plus or minus the individual values (of the unknowns) multiplied by an optional number deing severally (given), the sum (of the given quantities) divided by the number of unknowns increased or decreased by the multiplier will be the total value, thence the rest (can be determined).[1]

$$\Sigma x \pm cx_1 = a_1,\ \Sigma x \pm cx_2 = a_2,\ \Sigma x \pm cx_3 = a_3, \ldots\ldots$$
$$\Sigma x \pm cx \ = a_n$$

Therefore

$$\Sigma x = \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n \pm c}$$

Hence

$$x_1 = \frac{1}{c}\left(\pm a_1 \mp \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n \pm c} \right);$$

and so on for x_2, x_3 etc.

Now we shall give the rule enunciated by Brahmagupta for solving linear equations involving several unknowns:

> Removing the other unknowns from (the side of) the first unknown and dividing by the coefficient of the first unknown, the value of the first unknown (is obtained). In the case of more (values of the first unknown), *two and two (of them) should be considered* after reducing them to comon denominators. And (so on) repeatedly. If more unknowns remain (in the final equation), the method of the pulveriser (should be employed). (Then proceeding) reversely (the values of other unknowns can be found).[2]

Pṛthudaka Svāmī has commented on this rule as follows:

> In an example in which there are two or more unknown quantities, colours such as *yāvat-tāvat*, etc., should be assumed for their values. Upon them should

1. गच्छधनमिष्टगुणितैर्नैर्युतोनं पृथक् पृथक् सहितम् ।
गुणकयुतेन पदहृतं सर्वधनमतोऽवशेषाणि ॥ —*BrSpSi*. XIII. 47

2. आद्याद्वर्णादन्यान् वर्णान् प्रोह्याद्यमानमाद्यहृतम् ।
सदृशच्छेदासकृद् द्वौ द्वौ स्यातां कुट्टको बहुषु ॥ —*BrSpSi*. XVIII. 51

be performed, all operations conformably to the statement *of the example and thus should be carefully* framed two or more sides and also equations Equi-clearance should be made first between two and two of them and so on to the last. from one side one unknown should be cleared other unknowns reduced to a common denominator and also the absolute numbers should be cleared from the side opposite The residue of other unknowns being divided by the *residual coefficient of the first unknown will give the* value of the first unknown If there be obtained several such values. then with two and two of them equations should be formed after reduction to common denominators Proceeding in this way to the end find out the value of one unknown If that value be (in terms of) another unknown then the coefficients of those two will be reciprocally the values of the two *unknowns If however. there be present more un-*knowns in that value. the method of the pulveriser should be employed Arbitrary values may then be assumed for some of the unknowns

This problem would today be expressed in terms of the following equation :

$$S(t+x)=x\left\{s+\left(\frac{x-1}{2}\right)b\right\},$$

where x is the number of days after which the first overtakes the second. We may write this equation as

$$bx^2-\{2(S-s)+b\}x=2tS$$

whence the value x would be after solving the quadratic :

$$x=\frac{\sqrt{\{2(S-s)+b\}^2+8btS}+\{2(S-s)+b\}}{2b}$$

The *Bakhasali Manuscript* gives this solution as follows :

> The daily travel (S) diminished by the march of the first day (s) is doubled, this is increased by the common increment (b). That (sum) multiplied by itself is designated (as the *kṣepa* quantity). The product of the daily travel and the start (t) being multiplied by eight times the common increment, the *kṣepa* quantity is added. The square-root of this (is increased by the *kṣepa* quantity; the sum divided by twice the common increment will give the required number of days). (*BMS*. Folio 5. recto)

Aryabhata I (499 A D) is regarded as the founder of algebra, since he gives the solutions of a few quandratic problems For example, to find the number of terms of an arithmetical progression (A.P.), he gives the following rule :

> The sum of the series multiplied by eight times the common difference is added by the square of the difference between twice the first term and the common difference: the square-root (of the result) is diminished by twice the first term and (then) divided by the common difference half of this quotient plus unity is the number of terms [1]

In the modern notations of algebra, the solution would be expressed as follows :

1. गच्छोऽष्टोत्तरगुणिताद् द्विगुणाद्युत्तरविशेषवर्गयुतात् ।
मूलं द्विगुणाद्यूनं स्वोत्तरभजितं सरूपार्धम् ॥ —*Arya*. II. 20

$$n=\frac{1}{2}\left\{\frac{\sqrt{8bs+(2a-b)^2}-2a}{b}+1\right\}$$

There is another certain interest problem[1] the solution of which has been provided in the *Āryabhaṭīya* as

$$x=\frac{\sqrt{Apt+(p/2)^2}-p/2}{t}$$

which is the solution of the quadratic equation

$$tx^2+px-Ap=0$$

Ayabhaṭa I has thus given the solutions of a few quadratic equations but he nowhere gives the procedure of solving these equations

We give here the Rules of Brahmagupta for the solution of quadratic equations He undoubtedly is not the discoverer of these rules, but perhaps for the first time in the history of algebra we find the process of solving a quadratic equation so clearly indicated

First Rule

> The quadratic the absolute quantities multiplied by four times the coefficient of the square of the unknown are increased by the square of the coefficient of the middle (i e unknown), the square root of the result being diminished by the coefficient of the middle and divided by twice the coefficient of the square of the unknown is (the value of) the middle '[2]

This expressed in the modern notations would mean

$$x=\frac{\sqrt{4ac+b^2}-b}{2a}$$

It would be noted that in this rule Brahmagupta has employed the term *madhya* (middle) to imply the simple unknown as well as its coefficient The origin of the term' is doubtless connected with the mode of writing the quadratic equation in the form

$$ax^2+bx+0=0x^2+0x+c$$

so that there are three terms on each side of the equation

1 मूलफलं सफलं कालमूलगुणमर्धमूलकृतियुक्तम् । —*Ārya II* 25
मूलं मूलार्धोनं कालहृतं स्वमूलफलम् ॥

2 वर्गचतुर्गुणितानां रूपाणां मध्यवर्गसहितानाम् । —*BrSpSi* XVIII 44
मूलं मध्येनोनं वर्गद्विगुणोद्धृतं मध्यः ॥

Second Rule

> The absolute term multiplied by the coefficient of the square of the unknown is increased by the square of half the coefficient of the unknown the square root of the result diminished by half the coefficient of the unknown and divided by the coefficient of the square of the unknown is the unknown [2]

This when expressed in the modern algebraic notations would be

$$x=\frac{\sqrt{ac+(b/2)^2}-(b/2)}{a}$$

Here if the quadratic equation is

$$ax^2+bx+c=0$$

the absolute term is c (the one without the unknown x) the *coefficient of the square of unknown means the coefficient of* x^2 i e a and the coefficient of the unknown means the coefficient of x i e b

The above two methods of Brahmagupta are exactly the same as were suggested by Āryabhaṭa I

The root of the quadratic equation for the number of terms of an arithmetic progression (A P) is given by Brahmagupta according to the first rule [2]

$$n=\frac{\sqrt{8bs+(2a-b)^2}-(2a-b)}{2b}$$

Third Rule

> Brahmagupta also suggests a Third Rule which is very much the same as is used commonly now Though it has not been expressedly suggested as a new rule we find its application in a few instances For example this rule has been suggested in connection with the following problem on interest
>
> > A certain sum (p) is lent out for a period (t_1) the interest accrued (x) is lent out again at this

---

1 [illegible] ।
[illegible] ॥ —*BrSpSi* XVIII 45

2. [illegible] ।
[illegible] ॥ —*BrSpSi* XII 18

rate of interest for another period (t_2) and the total amount is A. Find x

The equation for determining x is

$$\frac{t_2}{pt_1}x^2+x=A$$

The solution of this equation would be

$$x=\sqrt{\left(\frac{pt_1}{2t_2}\right)^2+\frac{A\,pt_1}{t_2}}-\frac{pt_1}{2t_2}$$

Brahmagupta has stated the result in exactly the same form. Pṛthūdaka Svāmī has illustrated it in solving the following problem of interest

Problem

> A sum of five hundred *panas* (p) is lent out for a period of 4 months (t_1) the interest accrued (x) is lent out again at this rate of interest for another period of 10 months (t_2) and the total amount is 78 (A). Give the *pramāṇa-phala* i.e. the interest accrued x

Here *pramāṇa kāla* (t_1)=4 months

pramāṇa dhana (p)=500 panas

para kāla (t_2) the subsequent period=10 months

miśra dhana or the total interest accrued (A) =78 *paṇas*

Brahmagupta states his solution of such quadratics like this

> Take the product of the *pramāṇa dhana* (p) or the sum originally lent out and *pramāna kāla* i.e. the period for which originally lent out (t_1) and divide by the *para kāla* or the subsequent time (t_2) place this result at two places. Multiply the one placed at the first place with the *miśra dhana* (A) that is with the total interest accrued in this product add the square of half the one placed in the second place now take the square-root of it and from it subtract half of the one placed at the second place[1]

1 [illegible]
[illegible] —*BrSpSi* XII 15

Thus in the above example the product of *pramāṇa-dhana* and *pramāṇa kāla* divided by *parakāla* is (pt_1/t_2) is $\frac{500\times4}{10} = 200$. This is first multiplied by the total interest accrued (A); it becomes $200\times78=15600$. To this is now added square of half of 200 (which is 10000); it becomes 15600 plus $10000=25600$. Its square-root is taken which is 160. From this is subtacted half of the quantity (i e. half of 200 which is 100). Thus $160-100=60$. which is the answer. It was the interest which first accrued (x).

Another Quadratic Problem :

Brahmagupta refers to an astronomical problem which involves the quadratic equation

$$(72+a^2)x^2 \mp 24\,apx=144\left(\frac{R^2}{2}-p^2\right),$$

where a=*agra* (the sine of the amplitude of the Sun), b=*palabhā* (the equinoctial shadow of a gnomon 12 aṅguli long), R=radius, and x=*koṇaśaṅku* (sine of the altitude of the Sun when his altitude is 45°). Dividing out by $(72+a^2)$ we have

$$x^2 \mp 2mx=n,$$

where

$$m=\frac{12\,ap}{72+a^2},\; n=\frac{144(R^2/2-p^2)}{72+a^2}$$

Therefore we have

$$x=\sqrt{m^2+n}\pm m,$$

as stated by Brahmagupta. We find the same result in the *Sūrya-siddhānta* and in the text of Śrīpati. Āryabhaṭa II (1150) also followed the method of Āryabhaṭa I and Brahmagupta in solving a quadratic equation in connection with finding out the number of terms in an arithmetical progression (A.P.) whose first term is (a), common difference is b and the sum is s. The number of terms n is given by[1]

$$n=\frac{\sqrt{2bs+(a-b/2)^2}-a+b/2}{b}$$

Two Roots of a Quadratic Equation and Brahmagupta

A quadratic equation has two roots This must have been known to Indian algebraists even at a very early stage. Bhāskara II in his *Bijagaṇita* has quoted a rule ascribed to an ancient writer Padmanābha whose works are not available now :

1. *Mahāsiddhānta*, Bhāskara II, XV. 50

> If (after extracting roots) the square root of the absolute side (of the quadratic) be less than the negative absolute term on the other side, then taking it negative as well as positive two values (of the unknown) are found[1].

The term used here is *dvividhotpadyate mitih* which means that two values are obtained.

The existence of two roots of a quadratic equation appears to have been known also to Brahmagupta (628 A D) In illustration of his rules for the solution of a quadratic he has stated two problems involving practically the same equation .

> *Problem I* The square root of the residue of the revolution of the Sun less 2 is diminished by 1, multiplied by 10 and added by 2 when will this be equal to the residue of the revolution of the Sun less 1, on Wednesday ?[2]
>
> *Problem II* When will the square of one-fourth the residue of the exceeding months less three be equal to the residue of the exceeding months ?

We shall follow Pṛthūdaka Svāmi in solving the Problem I. In this problem the residue of the revolution of the Sun may be supposed to be x^2+2, then by the question, we have

$$10(x-1)+2=x^2+1,$$
$$\text{or } x^2-10x=-9$$

Again in Problem II, if we put $4x$ for the residue of the exceeding month then we have

$$(x-3)^2=4x$$

or $$x^2-10x=-9$$

Now by the second rule of Brahmagupta, retaining both the signs of the radical, we get .

$$x=5\pm\sqrt{25-9}=9 \text{ or } 1,$$

---

1 व्यक्त पक्षस्य चेन्मूलमन्यपक्षर्णरूपतः ।
अल्पं धनर्णगं कृत्वा द्विविधोत्पद्यते मितिः ॥ —Bhaskaka, *Bijaganita*

2 मण्डलशेषाद् द्व्यूनान्मूलं व्येकं दशाहतं द्वियुतम् ।
मण्डलशेषं व्येकं भानोर्बुधदिने कदा भवति ॥ —*BrSpSi* XVIII 49

3 अधिमासशेषपादात् त्र्यूनाद्वर्गोऽधिमासशेषसमः ।
अवमावशेषतो वावमशेषसमः कदा भवति ॥ —*BrSpSi*. XVIII. 50

As shown by Pṛthūdaka Svāmi the first value is taken by Brahmagupta for the Problem I and second value for the problem II Thus it is quite clear that Brahmagupta uses sometimes the positive and at other times the negative sign with the radical Hence we shall say that Brahmagupta knew that a quadratic equation would have two roots and according to the requisiteness of the problem one value out of the two would be utilised

Simultaneous Quadratic Equations

Indian authors usually treated problems involving various forms of simultaneous quadratic equations

(i) $\left.\begin{array}{l} x-y=d \\ xy=b \end{array}\right\}$ (ii) $\left.\begin{array}{l} x+y=a \\ xy=b \end{array}\right\}$

(iii) $\left.\begin{array}{r} x^2+y^2=c \\ xy=b \end{array}\right\}$ (iv) $\left.\begin{array}{r} x^2+y^2=c \\ x+y=a \end{array}\right\}$

For the solution of the combination (i) Āryabhaṭa I gives the following rule in his *Āryabhaṭīya*

> The square-root of four times the product (of two quantities) added with the square of their difference being added and diminished by their difference and halved gives the two multiplicands [1]

This means that

$$x=\tfrac{1}{2}(\sqrt{d^2+4b}+d),\quad y=\tfrac{1}{2}(\sqrt{d^2+4b}-d)$$

For the solution of the same combination Brahmagupta states as follows

> The square-root of the sum of the square of the difference of the residues and two squared times the product of the residues being added and subtracted by the difference of the residues and halved (gives) the desired residues severally [2]

(Here by difference of the residues is meant $x-y$; and by product of the residues is meant xy)

Brahmagupta does not seem to give the solution for simultaneous equations of the combination (ii) Mahāvīra (850 A D)

1 द्विकृतिगुणात् संवर्गाद् द्व्यन्तरवर्गेण संयुतान्मूलम् ।
अन्तरयुक्तं हीनं तद्गुणकारद्वयं दलितम् ॥ —*Ārya II 24*

2 शेषवधाद् द्विकृतिगुणात् शेषान्तरकृतियुतान्मूलम् ।
शेषान्तरेण युक्तं दलितं हीनं पृथक् शेषे ॥ —*BrSpSi XVIII 99*

has given the solution

> Subtract four times the area (of a rectangle) from the square of the semi perimeter then by *saṅkrumana* between the square-root of that (remainder) and the semi perimeter the base and the upright are obtained [1] (*GSS* VII 129½)

This expressed in the modern notations would be

$$x=\tfrac{1}{2}(a+\sqrt{a^2-4b})\quad y=\tfrac{1}{2}(a-\sqrt{a^2-4b})$$

For the combination (iii) Mahāvira in his *Gaṇita Sāra Saṃgraha* gives the following rule

> Add to and subtract twice the area (of a rectangle) from the square of the diagonal and extract the square roots By *saṅkramaṇa* between the greater and lesser of these (roots) the side and upright (are found) [2]

This put in modern notations would be

$$x=\tfrac{1}{2}(\sqrt{c+2b}+\sqrt{c-2b})$$
$$y=\tfrac{1}{2}(\sqrt{c+2b}-\sqrt{c-2b})$$

For the combination (iv) Aryabhata I gives the following rule

> *From the square of the sum (of two quantities)* subtract the sum of their squares Half of the remainder is their product [3]

The remaining operations will be similar to those for the equations (ii) so that

$$x=\tfrac{1}{2}(a+\sqrt{2c-a^2})\quad y=\tfrac{1}{2}(a-\sqrt{2c-a^2})$$

Brahmagupta in this connection says

> Subtract the square of the sum from twice the sum of squares, the square-root of the remainder being added to and subtracted from the sum and halved (gives) the desired residues [4]

1 *GSS* VII 129½

2 *GSS* VII 127½

3 सम्पर्कस्य हि वर्गाद्विशोधयेदेव वर्गसम्पर्कम् ।
यत्तस्य भवत्यर्धं विद्याद् गुणकारसंवर्गम् ॥ —*Āaya* II 23

4 कृति संयोगाद् द्विगुणाद्युति वर्ग शोध्य शेष मूल यत् ।
तेन युतोनो योगो दलित शेषे पृथगभीष्टे ॥ —*BrSpSi* XVIII 98

These equations have also been treated by Mahāvīra Bhāskara II and Nārāyaṇa. Nārāyaṇa has attempted two other forms of quadratic equations

$$\text{(v)}\quad \left.\begin{aligned} x^2+y^2&=c \\ x-y&=d \end{aligned}\right\} \qquad\qquad \text{(vi)}\quad \left.\begin{aligned} x^2-y^2&=m \\ xy&=b \end{aligned}\right\}$$

For their solutions, see Datta and Singh *Algebra* P. 84

Rule of Dissimilar Operations :

Datta and Singh say that the process of solving the following two particular cases of simultaneous quadratic equations was distinguished by most Indian mathematicians by the special designation *viṣama karma* or dissimilar operation

$$\text{(i)}\quad \left.\begin{aligned} x^2-y^2&=m \\ x-y&=n \end{aligned}\right\} \qquad\qquad \text{(ii)}\quad \left.\begin{aligned} x^2-y^2&=m \\ x+y&=p \end{aligned}\right\}$$

These equations have been regarded by these mathematicians as if of fundamental importance. They have given the following solutions (expressed in modern algebraic symbols)

For the combination (i)

$$x=\tfrac{1}{2}\left(\frac{m}{n}+n\right),\ y=\tfrac{1}{2}\left(\frac{m}{n}-n\right),$$

For the combination (ii)

$$x=\tfrac{1}{2}\left(p+\frac{m}{p}\right),\ y=\tfrac{1}{2}\left(p-\frac{m}{p}\right)$$

We shall express these solutions as follows in the words of Brahmagupta

> The difference of the squares (of the unknowns) is divided by the difference of the unknowns and the quotient is increased and diminished by the difference and divided by two (the results will be the two unknown quantities) (this is) dissimilar operation.

The same rule is restated by him on a different occasion in the course of solving a problem

> If then the differen ce of their squares, also the difference of them (are given) the difference of the squares

---

1 [illegible]
[illegible] *BrSpSi* XVIII 36

> is divided by the difference of them and this (latter) is added to and subtracted from the quotient and then divided by two (the results are) the residues whence the number of elapsed days (can be found)[1]

This *viṣama karma* or dissimilar operation has been described by other Indian algebraists also as Āryabhaṭa II (*Mahāsiddhānta* XVII 22) Śrīpati (*Siddhānta śekhara* XIV 13) Bhāskara II (*Līlāvatī*) and Nārāyana (*Gaṇita kaumudī* I 32)

Indeterminate Equations of the First Degree

Āryabhaṭa I should be given the credit of giving for the first time a treatment of the indeterminate equation of the first degree In his *Āryabhaṭīya* we find a method for obtaining the general solution in positive integers of the simple indeterminate equation

$$by-ax=c$$

for integral values of $a\ b\ c$ and further indicated how to extend it to get positive integral solutions of simultaneous indeterminate equations of the first degree His disciple Bhāskara I (522) showed that the same method might be applied to solve

$$by-ax=-c$$

and further that the solution of this equation would follow that of $by-ax=-1$ These methods of Āryabhaṭa I and Bhāskara I have also been adopted by Brahmagupta and in certain cases the improvement were suggested by Āryabhaṭa II in the middle of the tenth century A D

The problems which were treated by ancient Indian algebraists and which led them to the investigation of the simple indeterminate equation of the first degree may be classified under three heads

> Class I To find a number N which being divided by two given numbers $(a\ b)$ will leave two given remainders $(R_1\ R_2)$

Thus we have

$$N=ax+R_1=by+R_2$$

---

1 तद्वर्गान्तरभाजे तदन्तर चान्तरोद्धृतयुतोनम् ।
वर्गान्तर विभक्त द्वाभ्यां शेषे ततोद्युगण । *BrSpSi* XVIII 97

Hence $by - ax = R_1 - R_2$

Putting $\backsim R_2$

we get $by\text{-}ax = \pm c$

the upper or lower sign being taken according as R_1 is greater than or less than R_2.

Class II : To find a number (x) such that its product with a given number (α) being increased or decreased by another given number (δ) and then divided by a third given number (β) will leave no remainder.

This means that in other words, we shall have to get the solution of :

$$\frac{\alpha x \pm r}{\beta} = y$$

in positive integers.

Class III . Here we have to deal with an equation of the form :

$$by \dot{-} ax = \pm c$$

Kuṭṭaka, *Kuṭṭākāra* and *Kuṭṭa* : These are the three terms which Brahmagupta has used in regards to the subject of indeterminate analysis of the first degree. Āryabhaṭa I has also described this method in brief. but he does not use the word *kuṭṭaka*. In the *Mahābhāskarīya* of Bhāskara I we have the terms *kuṭṭakāra* and *kuṭṭa* (522 A.D) *MBh* I. 41,49). These words have been translated into English as *pulveriser* or *grinder*. Accordin g to Datta and Singh, the Hindu method of solving the equation $by\text{-}ax = \pm c$ is essentially based on a process of deriving from it *successively other similar equations in which the values of the* coefficients ($a\,b$) become smaller and smaller. Thus the process *is indeed the same as that of breaking a whole thing into smaller* pieces, and this accounts for its name *kuṭṭaka* or '*pulveriser*'.

In the problems of the Class I, the quantities (a and b) are called' divisors' *bhāgahāra*. *bhājaka*. *cheda* etc.) and R_1 and R_2 as 'remainders' (*agra* or *śeṣa* etc) while in a problem of the Class II , β is ordinarily called the 'divisor' (*bhāgahāra* or *bhājaka*) and γ the 'interpolator' *kṣepa*, *kṣepaka* etc.) ; here α is called the 'dividend' (*bhājya*), the unkown quantity to be found (x) is called the 'multiplier' or (*guṇaka* or *guṇakāra* etc) and y the

quotient or *phala* In later years Mahāvira has called the unknown number (x) as *rāśi*

Preliminary Operations in Kuttaka Karma

Usually it has been suggested that in order that an equation of the form

$$by-ax=\pm c \text{ or } by+ax=\pm c$$

may be amenable to solution the two numbers a and b must not have a common divisor for otherwise the equation would be absurd unless the number c had the same common divisor So before the rules which we shall give hereafter could be applied the numbers a b c must be made prime (*dṛḍha* or firm *niccheda* or having no divisor or *nirapavarta* meaning irreducible to each other

In this connection Bhāskara I writes

> The dividend and divisor will become prime to each other on being divided by the residue of their mutual division The operation of the pulveriser should be considered in relation to them [1]

Similarly we find in the writings of Brahmagupta

> Divide the multiplier and the divisor mutually and find the last residue those quantities being divided by *the residue will be prime to each other* [2]

Āryabhaṭa s Rule Āryabhata I is probably the first Indian writer on this subject but the operation given by him is rather obscure His disciple Bhāskara I has given the solution of indeterminate equations of the first degree in more satisfactory language We shall give here the translation of Āryabhaṭa s verse from the *Āryabhaṭiya* as rendered by Bibhutibhusan Datta because other translations of this verse do very often confuse the sense

> Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remain

1 भूमिच्छेदगुणौ योज्य भक्तशेषेण भाजितौ ।
हारभाज्यौ दृढौ स्यातां कुट्टाकारं तयोर्विदुः —*MBh* I 41

2 हृन्यौ परस्पर यच्छेषं गुणकारभागहारकयोः ।
तेन हृतौ निरच्छेदौ तावेव परस्परं हृन्यौ । —*BrSpSi* XVIII 9

der. The residue (and the divisor corresponding to the smaller remainder) being mutually divided, the last residue should be multiplied by such an optional integer that the product being added (in case the number of quotients of the mutual division is even) or subtracted (in case the number of quotients is odd) by the difference of the remainders (will be exactly divisible by the last but one remainder. Place the quotients of the mutual division successively one below the other in a column; below them the optional multiplier and underneath it the quotient just obtained). Any number below ... the penultimate) is multiplied by the one just above it and then added by that just below it. Divide the last number (obtained by doing so repeatedly) by the divisor corresponding to the smaller remainder; then multiply the residue by the divisor corresponding to the greater remainder and add the greater remainder. (The result will be) the number corresponding to the two divisors.[1]

There is an alternative rendering of this passage also as follows:

Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. The residue (and the divisor corresponding to the smaller remainder) being mutually divided (until the remainder becomes zero), the last quotient should be multiplied by an optional integer and then added (in case the number of quotients of the mutual division is even) or subtracted (in case the number of quotients *is odd) by the difference of the remainders. (Place the* other quotients of mutual division successively one below the other in a column; below them the result just obtained and underneath it the optional integer). Any

1 अधिकाग्रभागहारं छिन्द्यादूनाग्रभागहारेण ।
शेषपरस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तम् ॥
अध उपरि गुणितमन्त्ययुगूनाग्रच्छेदभाजिते शेषम् ।
अधिकाग्रच्छेदगुणं द्विच्छेदाग्रमधिकाग्रयुतम् । —*Ārya. II.* 32-33

number below (i e the penultimate) is multiplied by the one just above it and then added by that just below it. Divide the last number (obtained by doing so repeatedly) by the divisor corresponding to the smaller remainder, then multiply the residue by the divisor corresponding to the greater remainder and add the greater remainder (The result will be) the number corresponding to the two divisors

Āryabhaṭa's problem may be enunciated thus

To find a number (N) which being divided by two given numbers (a b) will leave two given remainders (R_1 R_2)

This gives

$$N=ax+R_1=by+R_2$$

(where R_1 is a greater remainder and R_2 lesser remainder, and a is the divisor corresponding to greater remainder and b the divisor corresponding to the lesser remainder)

Denoting as before by c the difference between R_1 and R_2, we get

(i) $by=ax+c$, if $R_1>R_2$

(ii) $ax=by+c$ if $R_2>R_1$

the equation being so written as to keep c always positive

Hence the problem now reduces to making either

$$\frac{ax+c}{b} \text{ or } \frac{by+c}{a}$$

according as $R_1>R_2$ or $R_2>R_1$ a positive integer So Āryabhaṭa says Divide the divisor corresponding to the greater remainder etc.

Now we shall proceed with the details of the operation as proposed by Datta and Singh in his *History of Hindu Mathematics Part II Algebra*

Suppose $R_1>R_2$ then the equation to be solved will be

$$ax+c=by \qquad (1)$$

a, b being prime to each other

Let

$$b)\ a\ (q$$
$$bq$$
$$\overline{}$$
$$r_1)\ b\ (q_1$$
$$r_1q_1$$
$$\overline{}$$
$$r_2)\ r_1\ (q_2$$
$$r_2q_2$$
$$\overline{}$$
$$r_3$$

$$\overline{}$$
$$r_{m-1})\ r_{m-2}\ (q_{m-1}$$
$$r_{m-1}q_{m-1}$$
$$\overline{}$$
$$r_m)\ r_{m-1}\ (q_m$$
$$r_mq_m$$
$$\overline{}$$
$$r_{m+1}$$

Then we get (when $a<b$ we shall have $q=0$ $r_1=a$)

$$a=bq+r_1$$
$$b=r_1q_1+r_2$$
$$r_1=r_2q_2+r_3$$
$$r_2=r_3q_3+r_4$$
$$\cdots$$
$$r_{m-2}=r_{m-1}\,q_{m-1}+r_m$$
$$r_{m-1}=r_mq_m+r_{m+1}$$

Now substituting the value of a in the given equation (1) we get

$$by=(bq+r_1)x+c$$

Therefore

$$y=qx+y_1$$

where

$$by_1=r_1x+c$$

In other words since $a=bq+r_1$ on putting

$$y=qx+y_1 \qquad \text{(ii)}$$

the given equation (i) reduces to

$$by_1=r_1x+c \qquad \text{(iii)}$$

Again since $b=r_1q_1+r_2$

putting similarly $x=q_1y_1+x_1$

the equation (iii) can be further reduced to

$$r_1x_1=r_2y_1-c \qquad \text{(iv)}$$

and so on

Writing down the successive values and reduced equations in columns we have

(1) $y=qx+y_1$	(I 1) $by_1=r_1x+c$
(2) $x=qy_1+x_1$	(I 2) $r_1x_1=r_2y_1-c$
(3) $y_1=q_2x_1+y_2$	(I 3) $r_2y_2=r_3x_1+c$
(4) $x_1=q_3y_2+x_2$	(I 4) $r_3x_2=r_4y_2-c$
(5) $y_2=q_4x_2+y_3$	(I 5) $r_4y_3=r_5x_2+c$
(6) $x_2=q_5y_3+x_3$	(I 6) $r_5x_3=r_6y_3-c$
(2n 1) $y_{n-1}=q_{2n-2}x_{n-1}+y_n$	(I 2n 1) $r_{2n-2}y_n=r_{2n-1}x_{n-1}+c$
(2n) $x_{n-1}=q_{2n-1}y_n+x_n$	(I 2n) $r_{2n-1}x_n=r_{2n}y_n-c$
(2n+1) $y_n=q_{2n}x_n+y_{n+1}$	(I 2n+1) $r_{2n}y_{n+1}=r_{2n+1}x_n+c$

Now the mutual division can be continued either (i) to the finish or (ii) so as to get a certain number of quotients and then stopped In either case the number of quotients found neglecting the first one (q) as is usual with Āryabhaṭa may be even or odd

Case (i) First suppose that the mutual division is continued until the zero remainder is obtained Since a b are prime to each other the last one remainder is unity

Subcase (i 1) Let the number of quotients be even We then have

$$r_{2n}=1 \quad r_{2n+1}=0 \quad q_{2n}=r_{2n-1}$$

The equations (I 2n) and (I 2n+1) therefore become

$$y_n=q_{2n}x_n+c$$

and

$$y_{n+1}=c$$

respectively Giving an arbitrary integral value (t) to x_n we get an integral value of y_n From that we can find the value of x_{n-1} by the equation (2n) Proceeding backwards step by step we ultimately find the values of x and y in positive integers So that the equation (I) is solved

Subcase (i 2) If the number of quotients be odd we shall have

$$r_{2n-1}=1 \quad r_{2n}=0 \quad q_{2n-1}=r_{2n-2}$$

The equations $(2n+1)$ and $(I\ 2n+1)$ will then be absent and the equations $(I\ 2n-1)$ and $(I\ 2n)$ will be reduced respectively to

$$x_{n-1}=q_{2n-1}\,y-c$$

and $x_n=-c$

Giving an arbitrary integral value (t) to y_n we get an integral value of x_{n-1} Then proceeding backwards as before we calculate the values of x and y

Case (ii) Next suppose that the mutual division is stopped after having obtained an even or odd number of quotients

Subcase (ii.1) If the number of quotients obtained be *even* the reduced form of the original equation is

$$r_2\ y_{n+1}=r_{2n+1}\,x_n+c$$

or $$y_{n+1}=\frac{r_{2n+1}\,x_n+c}{r_2}$$

Giving a suitable integral value (t) to x_n as will make

$$y_{n+1}=\frac{r_{2n+1}\,t+c}{r_{2n}}=\text{an integral number.}$$

we get an integral value for y_n by $(2n+1)$ The values of x and y can then be calculated by proceeding as before

Subcase (ii 2) If the number of quotients be *odd* the reduced form of the quotient is

$$r_{2n-1}\,x_n=r_{2n}\,y_n-c$$

or $$x_n=\frac{r_{2n}\,y_n-c}{r_{2n-1}}$$

Putting $y_n=t$ where t is an integer such that

$$x_n=\frac{r_{2n}\,t-c}{r_{2n-1}}=\text{a whole number}$$

we get an integral value of x_{n-1} by $(2n)$ Whence can be calculated the values of x and y in integers

If $x=\alpha$ and $y=\beta$ be the least integral solution of $ax+c=by$, we shall have

$$a\alpha+c=b\beta$$

Therefore $a(bm+\alpha)+c=b\,(am+\beta)$

m being any integer Therefore in general

$$x=bm+\alpha$$

But we have calculated before that

$$x = q_1 y_1 + x_1$$

$$q_1 y_1 + x_1 = bm + \alpha$$

Thus it is found that the minimum value α of x is equal to the remainder left on dividing its calculated value by b whence we can calculate the minimum value of N $(= a\alpha + R_1)$ This will explain the *rationale* of the operations described in the latter portion of the rule of Āryabhaṭa I

Bhāskara I and Kuttaka Operation

In Chapter I of the *Mahābhāskarīya* Bhāskara I has described the preliminary operation to be performed on the divisor and dividend of a pulveriser We shall quote it from the edition of K S Shukla

> The divisor (which is the number of civil days in a *yuga*) and the dividend (which is the revolution number of the desired planet) become prime to each other on being divided by the (last non zero) residue of the mutual division of the number of civil days in a *yuga* and the revolution number of the desired planet The operations of the pulveriser should be performed on them (i e on the abraded divisor and abraded dividend) So has been said [1]

An indeterminate equation of the first degree of the type

$$\frac{ax - c}{a} = y$$

(with x and y unknown) is known in Hindu mathematics by the name of pulveriser —*kuṭṭākāra*) In this equation a is called the dividend' (*bhājya*) b the divisor' (*bhāgahāra*) c the interpolator (*kṣepa*) x the multiplier (*guṇakāra*) and y the "quotient (*labdha*)

In the pulveriser contemplated in the above stanza

a = revolution number of a planet

b = civil days in a *yuga*

c = residue of the revolutions of the planet (*Śeṣa*)

1 भूदिनेभगणान्योन्य भक्तशेषेण भाजितौ ।
हारभाज्यौ दृढौ स्यातां कुट्टाकारस्तयोरिह । —*MBh* I 41

x=*ahargana*,

and y=complete revolutions performed by the planet

The text says that as a preliminary operation to the solution of this pulveriser, a and b i e, civil days in *yuga* and revolution number of the planet should be made prime to each other by dividing them out by their greatest common factor, That is to say, in solving a pulveriser, one should always make use of abraded divisor and abraded dividend

The interpolator, i e, the residue should also be divided out by the same factor (This instruction is not given in the text but it is implied that the residue should be computed for the abraded dividend and abraded divisor)

> Set down the dividend above and the divisor (*hara*) below that Divide them mutually and write down the quotients (*labdha*) of division one below the other (in the form of a chain) (When an even number of quotients is obtained) think out by what number the (last) remainder be multiplied so that the product being diminished by the (given) residue be exactly divisible (by the divisor corresponding to that remainder) Put down the chosen number called *mati* below the chain and then the new quotient underneath it Then by the chosen number multiply the number which stands just above it and to the product add the quotient (written below the chosen number) (Replace the upper number by the resulting sum and cancel the number below) Proceed afterwards also in the same way (until only two numbers remain) Divide the upper number (called the "multiplier) by the divisor by the usual process and the lower one (called the quotient) by the dividend the remainders (thus obtained) will respectively be the *ahargaṇa* and the revolutions etc or what one wants to know[1]

We shall illustrate the operation by taking a problem from the *Laghu Bhāskarīya* (VIII 17)

> The sum the difference and the product increased by one of the residues of the revolution of Saturn and Mars—each is a perfect square Taking the equations

furnished by the above and applying the method of such quadratics obtain the (simplest) solution by the substitution of 2, 3 etc successively in the general solution) Then calculate the *ahragana* and the revolutions performed by Saturn and Mars in that time together with the number of solar years elapsed [2]

Let x and y denote the residues of the revolution of Mars and Saturn respectively Then we have to find out two numbers x and y such that each of the expressions $x+y$ $x-y$ and $xy+1$ may be a perfect square

Let $x+y=4P^2$ and $x-y=4Q^2$ so that

$$x=2P^2+2Q^2$$

$$y=2P^2-2Q^2$$

and therefore $xy+1=(2P^2-1)^2+4(P^4-Q^4)$

Hence the condition that $xy+1$ be a perfect square is that $P^2=Q^4$ Substituting these values we have

$$x=2(Q^4+Q^2)$$

$$y=2(Q^4-Q^2)$$

where Q may possess any of the values 2, 3 4 but not 1 (We neglect the case when x or y is zero)

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1 भाज्यं न्यसेदुपरि हारमधश्च तस्य ।
खण्ड्यात् परस्परमधो विनिधाय लब्धम् ।
केनाऽऽहतोऽयमपनीय यथाऽस्य शेषं ।
भागं ददाति परिशुद्धमिति प्रचिन्त्यम् ॥

आप्तां मतिं तां विनिधाय वल्ल्यां
नित्यं ह्यधोऽधः क्रमशश्च लब्धम् ।
मत्या हतं स्यादुपरिस्थितं य
ल्लब्धेन युक्तं परतश्च तद्वत् ॥

हारेणभाज्यो विधिनोपरिस्थो
भाज्येन नित्यं तदधः स्थितश्च ।
अहर्गणोऽस्मिन् भगणाश्च पूर्वं
तदा भवेद्यस्य समीहितं यत् ॥ —*MBh* I, 42-44

2 शेषौ मण्डलजौ यमक्षितिजयोः संयुक्तविश्लेषितौ
अन्योन्याहतवर्गितौ च पददौ रूपेण युक्तौ यदि ।
एवं साधु विचिन्त्य गणितविदा द्विनिघ्नमाद्वत्सरै ।
संगण्या द्युगणादयश्च सुना कालेन कालोद्भवा ॥ —*LBh* VIII 17

Putting Q=2, we get $x=40$ and $y=24$ which is the least solution

Assuming now that the residues of the revolution (*maṇḍalaja śeṣa*) of Saturn and Mars are 24 and 40 respectively we have to obtain the *ahargaṇa* (which means the number of mean civil days elapsed since the beginning of Kaliyuga or, in fact, any epoch)

The revolution-number of Saturn is 146564 and the number of civil days in a *yuga* is 1,577 917 500 In the present problem, these are respectively the dividend and the divisor Their H C F is 4, so that dividing them out by 4 we get 36641 and 394 479,375 as the abraded dividend and abraded divisor respectively We have, therefore, to solve the pulveriser

$$\frac{36641x-24}{394479375}=y$$

where x and y denote the *ahargaṇa* and the revolutions respectively made by Saturn

Mutually dividing 36641 and 394479375 we get

36641) 394479375 (10766

394477006

2369) 36641 (15

35535

1106) 2369 (2

[illegible]

157) 1106 (7

1099

7) 157 (22

154

3) 7 (2

6

$1\times27-24=3$)3(1

3

0

We have chosen here the number 27 as the optional number (*mati*) In fact, *mati* may be chosen at any stage after an even number of quotients are obtained

Writing down the quotients one below the other as prescribed in the rule, we get the chain

10766
15
2
7
22
2
(*mati*) 27
1

Reducing the chain we successively get

10766	10766	10766	10766	10766	10766	3108044439 (multiplier)
15	15	15	15	15	288689	288689 (quotient)
2	2	2	2	18665	18665	
7	7	7	8714	8714		
22	22	1237	1237			
2	55	55				
(*mati*) 27	27					
1 = c_3						

(it would be seen in this reduction of chain that *mati* or 27×2 plus 1 is 55 55×22 plus 27 is 1237, 1237×7 plus 55 is 8714 8714×2 plus 1237 is 18665, 18665×15 plus 8714 is 288689 and finally 288689×10766 plus 18665 is 3108044439 which is the multiplier)

Dividing 3108044439 by 394479375 and 288689 by 36641, we obtain 346688814 and 32202 respectively as remainders. (This division is performed only when the multiplier and quotient are greater than the divisor and dividend respectively) These are the minimum values of x and y satisfying the above equation

Therefore the required *ahargaṇa* = 346688814 and the revolutions performed by Saturn = 32202

To obtain the *ahargaṇa* and the revolutions of Mars, one has to solve the equation

$$\frac{191402\, x - 40}{131493125} = y$$

Brahmagupta further observes

> *Such is the process when the quotients (of mutual division) are even in number But if they be odd what has been stated before as negative should be made as positive or as positive should be made negative*[1]

Regarding the direction for dividing the divisor corresponding to the greater number by the divisor corresponding to the smaller remainder Prthudaka Svami (860 A D) observes that it is not absolute rather optional so that the process may be conducted in the same way by starting with the division of the divisor corresponding to the smaller remainder by the divisor corresponding to the greater remainder But in this case of inversion of the process he continues the difference of the remainders must be negative

That is to say the equation

$$by=ax+c$$

can be solved by transforming it first to the form

$$ax=by-c$$

so that we shall have to start with the division of b by a

For the details of the "Theory of the pulveriser" as applied to the problems in Astronomy the reader is referred to the writings of Bhatta Govind translated by K S Shukla and given as an Appendix to the edition of the *Laghu Bhaskariya* For the *rationale* of the rules in relation to *kuttaka* or the pulveriser operation one may also refer to the chapters by Datta and Singh in the *History of Hindu Mathematics Algebra*

Solution of $by=ax\pm 1$

This simple indeterminate equation has a special use in astronomical calculations and therefore Indian algebraists have paid special attention to it In fact this equation is solved exactly in the same way as the equation $by=ax\pm c$ it is a parti

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1 एवं समेषु विषमेष्वृणं धनं धनमृणं यदुक्तं तत् ।
ऋणधनयोर्व्यस्तं गुणप्रक्षेपौ कार्यम् ॥ —*BrSpSi* XVIII 13

cular case only of the more general latter equation. Of course, there is a little justification also for treating it separately since both the types of equations represent two different physical conditions of the astronomical problems. In the case of $by=ax\pm c$, the conditions are such that the value of either y or x more particularly of the latter, has to be found and the rules for solution formulated with that objective. But in the case of the equation $by=ax\pm 1$ the physical conditions require the values of both y and x.

The equation $by=ax\pm 1$ is usually known by the name *sthira kuṭṭaka* literally meaning the *constant pulveriser*. Pṛthūdaka Svāmi also names it as *dṛḍha kuṭṭaka* meaning firm-pulveriser. Later on this term *dṛḍha* was confined to another sense equivlent to *nicched* (having no divisor) or *nirapavarta* (irreducible). The origin of the name *sthira kuṭṭaka* or constant pulveriser has been explained by Pṛthūdaka Svāmi as being due to the fact that the interpolator (± 1) is here invariable.

For the solution of this equation we shall quote Bhāskara I's rule and the rule by Brahmagupta. Bhāskara I writes in this connection as follows:

> The method of the pulveriser is applied also after subtracting unity. The multiplier and quotient are respectively the numbers above and underneath. Multiplying those quantities by the desired number divide by the reduced divisor and dividend, the residues are in this case known to be the (elapsed) days and (residues of) revolutions respectively.[1]

The pulveriser

$$\frac{ax-c}{b}=y \qquad (1)$$

may be written as

$$\frac{a\lambda-1}{b}=Y \qquad (2)$$

where $x=c\lambda$ and $y=cY$. If $\lambda=\alpha$, $Y=\beta$ is a solution of (2), then $x=c\alpha$, $y=c\beta$ will be a solution of (1). Hence the above rule.

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1 [illegible] ।
[illegible] ॥ —*MBh* I 45

If the multiplier be negative it must be made positive and the additive must be made negative and then the *method of the pulveriser should be employed*

Pṛthūdaka Svāmī however, *does not indicate how to derive* the solution of the equation

$$by = -ax + c \qquad (1)$$

from that of the equation

$$by = ax - c \qquad (2)$$

The method however seems to have been this

Let $x = \alpha$ $y = \beta$ be the minimum solution of (2) Then we get

$$b\beta = a\,\alpha - c$$

or $\qquad b(a\ \beta) = a(\alpha - b) + c$

Hence $x = \alpha - b$ $y = a - \beta$ is the minimum solution of (1) This rule is very clearly indicated by Bhāskara II and others

We shall give two examples from Bhāskara II (*Bijagaṇita*) to illustrate the rule

Example I

$13y = -60x + 3$

By the method described before we find that the minimum solution of

$$13y = 60x + 3$$

is $x = 11$ $y = 51$ Subtracting these values from their respective abraders, namely 13 and 60 we get 2 and 90 Then by the maxim "In the case of the dividend and divisor being of different signs, the results from the operation of division should be known to be so making the quotient negative we get the solution of

$$13y = -60x + 3$$

as $x = 2$, $y = -9$ Subtracting these values again from their respective abraders (13 60), we get the solution of

$13y = -60x - 3$

as $x = 11$ $y = -51$

Example II

$11y = 18x + 10$

Proceeding as before, we find the minimum solution of

$$11y = 18x + 10$$

to be $x=8$, $y=14$ These will also be the values of x and y in the case of the negative divisor but the quotient for the reasons stated before should be made negative So the solution of

$$-11y = 18x + 10$$

is $x=8$, $y=-14$ Subtracting these (i e, their numerical values) from their respective abraders, we get the solution of

$$-11y = 18x - 10$$

as $x=3$, $y=-4$

> "When the divisor is positive or negative the numerical values of the quotient and multiplier remain the same when either the divisor or the dividend is negative, the quotient must always be known to be negative"[1]

One Linear Equation in More Than Two Unknowns

Whenever a linear equation involves more than two unknown's the Indian algebraists used to assume arbitrary values for all the unknowns except two and then to apply the method of *kut taka* or "pulveriser' In this connection, Brahmagupta says.

> The method of the pulveriser (should be employed if there be present many unknowns (in any equation)[1],

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1 Bhāskara II gives the following rule

"Those(the multiplier and quotient)obtained for a positive dividend being treated in the same manner give the results corresponding to a negative dividend"

The treatment alluded to in this rule is that of subtraction from the respective abraders He has further elaborated it thus

The multiplier and quotient should be determined by taking the dividend, divisor and interpolator as positive They will be the quantities for the additive interpolator Subtracting them from their respective abraders the quantities for a negative interpolator are found If the dividend or divisor be negative the quotient should be stated as negative the quotient should be stated as negative

—*Bijaganita*

1. [illegible]

[illegible] —*BrSpSi* XVIII 51

We shall take up one of the problems posed by Brahmagupta concerning astronomy and leading to the equation [1]

$$197x - 1644y - z = 6302$$

Hence

$$x = \frac{1644y + z + 6302}{197}$$

The commentator assumes $z = 131$ Then

$$x = \frac{1644y + 6433}{197}$$

hence by the usual method of the pulveriser

$$x = 41 \quad y = 1$$

General Problem of Remainders

A certain type of simultaneous indeterminate equations of the first degree arise out of the general problem of remainders which may thus be stated To find a number N which being severally divided by a_1 a_2 a_3 a_n leaves as remainders r_1 r_2 r_3 r_n respectively

While dealing with such a case we shall have the following series of equations

$$N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \quad = a_n x_n + r$$

We have reasons to believe that the method of solution of these equations was known to Āryabhaṭa I In the translation of the verse in the *Āryabhaṭīya* II 32 33 (the translation of which we have already given) the term *dvicchedāgram* should be translated as the result will be the remainder corresponding to the product of the two divisors instead of the result will be the number corresponding to the two divisors (the last line of the translation) This explanation is in fact given by Bhāskara I the direct disciple and earliest commentator of Āryabhaṭa I Such a rule is clearly stated by Brahmagupta[2]

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1 [illegible] ।
[illegible] ॥ —*BrSpSi* XVIII 55

2 [illegible] ।
[illegible] ॥ —*BrSpSi* XVIII 5

The rationale of this method is not difficult I shall quote it from the book of Datta and Singh Starting with the consideration of the first two divisors we have

$$N=a_1x_1+r_1=a_2x_2+r_2$$

By the method described before we can find the minimum value α of x_1 satisfying this equation Then the minimum value of N will be $a_1\alpha+r_1$ Hence the general value of N will be given by

$$N=a_1\ (a_2t+\alpha)+r_1$$
$$=a_1a_2\ t+a_1\alpha+r_1$$

where t is an integer Thus $a_1\alpha+r_1$ is the remainder left on dividing N by $a_1\,a_2$ as stated by Ārybhaṭa I and Brahmagupta Now taking into consideration the third condition we have

$$N=a_1a_2t+a_1\alpha r_1=a_3x_3+r_3$$

which can be solved in the same way as before Proceeding in this way successively we shall ultimately arrive at a value of N satisfying all the conditions

Pṛthūdaka Svāmi remarks

> Wherever the reduction of two divisors by a common *measure is possible there the product of the divisors'* should be understood as equivalent to the product of the divisor corresponding to the greater remainder and quotient of the divisor corresponding to the smaller remainder as reduced (i e. divided) by the common measure [1] *When one divisor is exactly divisible by the* other then the greater remainder is the (required) remainder and the divisor corresponding to the greater remainder is taken as the product of the divisors (The truth of) this may be investigated by an intelligent mathematician by taking several symbols

As an illustration we shall take up a problem quoted by Bhāskara II in his *Bijagaṇita* and which in its solution follows the method of Āryabhaṭa I Pṛthūdaka Svāmi while commenting on serveral verses from Brahmagupta (*BrSpSi* XVIII 3–6)

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1 i e if p be the L.C.M of a and a_2 the general value of N satisfying the above two conditions will be

$$N=pt+a_1\alpha+r$$

instead of $\quad N=a_1a_2t+a_1\alpha+r_1$

observes that such problems were very popular amongst the ancient Indian mathematicians

Problem To find a number N which leaves remainders 5 4 3 2 when divided by 6,5,4,3 respectively

That is to solve the equations

$$N=6x+5=5y+4=4z+3=3w+2$$

We have since $N=6x+5=5y+4$,

$$x=\frac{5y-1}{6}$$

But x must be integral, so $y=6t+5$, $x=5t+4$

Hence $N=30t+29$

Again $N=30t+29=4z+3$

Therefore $t=\dfrac{2z-13}{15}$

Since t must be integral we must have $z=15s+14$ hence $t=2s+1$ Therefore

$$N=60s+59$$

The last condition is identically satisfied The method given here is the one followed by Pṛthūdaka Svāmi

Thus when $N=60s+59=6x+5$

$$x=\frac{60s+54}{6}=10s+9 \qquad (1)$$

Again when $N=60s+59=5y+4$,

$$y=\frac{60s+55}{5}=12s+11$$

Again when $N=60s+59=4z+3$

$$z=\frac{60s+56}{4}=15s+14$$

Lastly when $N=60s+59=3w+2$,

$$w=\frac{60s+57}{3}=20s+19$$

Varga Prakṛti or Kṛti Prakṛti or Square-Nature

The word *varga-prakṛti* (literally meaning 'square-nature') has been given by Indian algebraists to the indeterminate quadratic equation

$$Nx^2 \pm c = y^2$$

Here in this equation the absolute number c should be *rūpa* (or unity) which *means the equation*

$$Nx^2 \pm 1 = y^2$$

or it may be any absolute number The most fundamental equation of this class has been regarded as

$$Nx^2 + 1 = y^2$$

where N *is a non square integer*

This branch *of mathematics has originated from the number* which is the *prakṛti* of the square of *yāvat*, etc. (the unknown x etc) and therefore it is called *varga prakṛti The quantity* N of *the above equation is known as Prakṛti* Brahmagupta uses the term GUNAKA (*multiplier*) for the same purpose[1]

This term *guṇaka* together with its variation *guṇa* appears *occasionally also in the writings of later authors For example*, Śrīpati (*Siddhānta śekhara* XIV 32) employs the term *guṇaka* where as Bhāskara II and Nārāyana use the term *guṇa* in their *Bījagaṇitas*

In this connection we would now like to quote from Pṛthūdaka Svāmī (863 A D) from his commentary on the *Brāhmasphuṭasiddhānta*

> Here are stated *for ordinary use the terms* which are well known to people The number whose square multiplied by an optional multiplier and then increased or decreased by another optional number becomes capable of yielding a square root *is designated by the* term the "lesser root *kaniṣṭha pada* or the "first root' *ādya-mūla*) The root which results after those operations have been performed is called by the name the "greater root (*jyeṣṭha pada*) *or the* 'second root' (*anya mūla*) *If there* be a number multiplying both these roots, it is called the augmenter (*udvartaka*), and on the contrary, if there be a number dividing the roots it is called the abridger" (*apavartaka*).

Thus in the equation

$$Nx^2 \pm c = y^2.$$

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1 मूलं द्विधेष्ट वर्गाद् गुणक गुणादिष्टयुत विहीनाच्च ।
आद्यवधो गुणकगुण सहान्त्यघातेन कृतमन्त्यम् ॥ —*BrSpSi* XVIII 64

2 *BrSpSi XVIII* 64 (Com.)

x is known as the lesser root, y is the greater root, N is the multiplier (*guṇaka*) and c is interpolator or *kṣepaka*. Bhāskara II has used the word "*hrasvamūla*" for *kaniṣṭha pada* or *ādya-mūla* literally meaning "lsser root". The earlier terms, the "first root" (*ādyamūla*) for the value of x and the "second root" or the "last root" *antya-mūla* for the value of y are quite free from ambiguity Their use is found in the algebra of Brahmagupta. The later terms appears in the works of his commentator Pṛthūdaka Svāmī.

Brahmagupta uses the term *kṣepa*, *prakṣepa* or *prakṣepaka* in the sense of "interpolator." Again, when negative, the interpolator is sometimes distinguished as the "subtractive" or *śodhaka* and the positive interpolator is then called "the additive."

Lemmas of Brahmagupta

Prior to our giving the general solution of the Square-nature or *Varga-Prakṛti*, it would be better to give two Lemmas established by Brahmagupta We have the following in the *Brāhmasphuṭasiddhānta* :

> Of the square of the optional number multiplied by the *guṇaka* and increased or decreased by an other optional number, *iṣṭa*, (extract) the square root. (Proceed) twice. The product of the first roots multiplied by the *guṇaka* together with the product of the second roots will give a (fresh) second root; the sum of their cross-products will be a (fresh) first root. The (corresponding) interpolator will be equal to the product of the (previous) interpolators[1]

There is a little difficulty in ascertaing the real sense of the rule given in these lines since the word *dvidha* (twice) has two implications Firstly, it may mean that the earlier operations of finding roots are made on two optional numbers with two optional interpolators, and with the results thus obtained the subse-

1. मूलं द्विधेष्टवर्गाद् गुणकगुणादिष्टयुतविहीनाच्च ।
आद्यवधो गुणकगुणः सहान्त्यघातेन कृतमन्त्यम् ॥
वज्रवधैक्यं प्रथमं प्रक्षेपः क्षेपवधतुल्यः ।
प्रक्षेपशोधकहृते मूले प्रक्षेपके रूपे ॥ —*BrSpSi.* XVIII 64-65

quent operations of their composition are performed. Secondly, it may also mean that the earlier operations are made with one optionally chosen number and one interpolator, and the subsequent ones are carried out after the repeated statement of those roots for the second time. It is also implied that in the composition of the quadratic roots, their products may be added together or subtracted from each other.

In other words, if $x=\alpha$, $y=\beta$ be a solution of the equation :

$$Nx^2+k=y^2,$$

and $x=\alpha'$, $y=\beta'$ be a solution of

$$Nx^2+k'=y^2,$$

then according to the above

$$x=\alpha\beta'\pm\alpha'\beta,\ y=\beta\beta'\pm N\alpha\alpha'$$

is a solution of the equation

$$Nx^2+kk'=y^2.$$

In other words, if

$$N\alpha^2+k=\beta^2$$
$$N\alpha'^2+k'=\beta'^2$$

then

$$N(\alpha\beta'\pm\alpha'\beta)^2+kk'=(\beta\beta'\pm N\alpha\alpha')^2 \qquad \text{(I)}$$

In particular, taking $\alpha=\alpha'$, $\beta=\beta'$ and $k=k'$, Brahmagupta finds from a solution $x=\alpha$, $y=\beta$ of the equation

$$Nx^2+k^2=y^2$$

a solution $x=2\alpha\beta$, $y=\beta^2+N\alpha^2$ of the equation

$$Nx^2+k=y^2$$

That is, if

$$N\alpha^2+k=\beta^2$$

We can compose this solution with the previous ones, and get another solution and thus proceed on to innumerable solutions From *Brahmagupta's Corollary to First Lemma* we get another set of solutions If $(a\ b)$ be solution of the Square-nature then another solution of it is

$$x=2ab \text{ and } y=b^2+Na^2$$

Thus even if we have only one solution we can get the other solution also (since N is known), and thus we can get any number of solutions one after the other by this Principle of Composition

Brahmagupta's Lemmas have been described by Bhāskara II (1150 A D) in the following words

> Set down successively the lesser root (*hrasva*) greater root (*jyeṣṭha*) and interpolator (*kṣepaka*) and below them should be set down in order the same or an another (set of similar quantities) From them by the Principle of Composition (Bhāvanā) can be obtained numerous roots Therefore the Principle of Composition will be explained here (Find) the two cross products (*vajrābhyāsa*) of the two lesser and the two greater roots their sum is a lesser root Add the product of the two lesser roots multiplied by the *prakṛti* to the product of the two greater roots the sum will be a greater root In that (equation) the interpolator will be the product of the two previous interpolators Again the difference of the two cross products is a lesser root Subtract the product of the two lesser roots multiplied by the *prakṛti* from the product of the two greater roots (the difference) will be greater root Here also the interpolator is the product of the two (previous) interpolators [1]

---

1 [illegible] ।
[illegible] ॥
[illegible] ।
[illegible] ॥
[illegible] ।
[illegible] ।

Bhāskara II *Bījagaṇita VargaPrakṛti* 2 4

Principle of Composition

The above results have been technically known amongst Indian algebraists as *Bhāvanā* (demonstrated or proved, hence theorem or lemma) The word *bhāvanā* also means "composition or combination" in algebra Bhāvanā may be of two types *Samāsa Bhāvanā* (or addition Lemma, or additive composition) and *Antara Bhāvanā* (or subtraction Lemma or subtractive composition). *Whenever, again the bhāvanā is made with two* equal sets of roots and interpolators it is technically named as *Tulya Bhāvanā* (or composition of equals) and when with two unequal sets of values then it is known as *Atulya Bhāvanā* (or composition of unequals)

Proof of Brahmagupta's Lemmas

It is significant to be indicated that Brahmagupta's Lemmas were rediscovered by Euler in *1764* and by *Lagrange* in *1768*, and a considerable importance was attached to them Kṛṣṇa (1580 A D) the commentator on the *Bījagaṇita* of Bhāskara II gives the following proof of Brahmagupta's Lemmas :

Let (α,β) and (α',β') be the two solutions of the equation $nx^2+k=y^2$

Brahmagupta's Corollary also follows at once from the above by putting $\alpha'=\alpha$, $\beta'=\beta$ and $k'=k$.

$$N\,(2\alpha\beta)^2+k^2=(\beta^2\pm N\alpha^2)^2$$

Thus the roots are $x=2\alpha\beta$ and $y=\beta^2\pm N\alpha^2$ which is the Corollary.

It would be seen that modern historians of mathematics are incorrect when they say that Fermat (1657) was the first to state that the equation $Nx^2+1=y^2$, where N is a non-square integer has an unlimited number of solutions in integers. For this assertion, history takes us to the early Seventh Century A.D. when Brahmagupta wrote his classical treatise, the *Brāhmasphuṭasiddhānta*, and gave the well known two Lemmas and the Corollary to the first Lemma.

Second Lemma of Brahmagupta

In the *Brāhmasphuṭa siddhānta*, we find another important Lemma by Brahmagupta stated as follows :

> On dividing the two roots (of a square-Nature) by the square-root of its additive or subtracrive, the roots for interpolator unity (will be found).[1]

This Lemma when expressed in the modern language of algebra would mean that if $x=\alpha, y=\beta$ be a solution of the equation.

$$Nx^2+k^2=y^2$$

then $x=\alpha/k$, $y=\beta/k$ is a solution of the equation

$$Nx^2+1=y^2.$$

This rule, at another place, has been re-enunciated as follows :

> If the interpolator is that divided by a square then the roots will be those multiplied by its square-root.[2]

---

1. क्षेपे पदोद्धृते मूले क्षेपे रूपे ।

—*BrSpSi*. XVIII. 65

2. वर्गच्छिन्ने क्षेपे क्षेपपदगुणिते तदा मूले ।

—*BrSpSi*. XVIII. 70

This rule may be expressed in terms of symbols as follows Suppose the *Varga prakṛti* (Square-nature) to be

$$Nx^2 \pm p^2 d = y^2,$$

so that its interpolator (*kṣepa*) $p^2 d$ is exactly divisible by the square p^2. Then putting therein $u = x/p$, $v = y/p$, we derive the equation

$$Nu^2 \pm d = v^2$$

whose interpolator is equal to that of the original Square-nature divided by p^2 It is clear that the roots of the original equation are p times those of the derived equation

Rational Solution

Indian algebraists have usually suggested the following method to obtain a first solution of $Nx^2 + 1 = y^2$

> Take an arbitrary small rational number, α, such that its square multiplied by the *guṇaka* N and increased or diminished by a suitably chosen rational number k will be an exact square.

In other words, we shall have to obtain empirically a relation of the form

$$N\alpha^2 \pm k = \beta^2$$

where α, k, and β are rational numbers Let us call this relation as the *Auxiliary Equation* Then by Brahmagupta's Corollary, we get from it the relation

$$N(2\alpha\beta)^2 + k^2 = (\beta^2 + N\alpha^2)^2,$$

or

$$N\left(\frac{2\alpha\beta}{k}\right)^2 + 1 = \left(\frac{\beta^2 + N\alpha^2}{k}\right)^2$$

Hence, one rational solution of the equation $Nx^2 + 1 = y^2$ is given by

$$x = \frac{2\alpha\beta}{k}, \quad y = \frac{\beta^2 + N\alpha^2}{k}$$

Work on the rational solution of the Square-nature has been also done by Śrīpati In fact his solution, given in 1039 A D is of historical significance. He derives the rational solution without the aid of the "auxiliary equation" He gives the following rule

> Unity is the lesser root. Its square multiplied by the *prakṛti* is increased or decreased by the *prakṛti* combined with an (optional) number whose square-root will be the greater root. From them will be obtained two roots by the Principle of Composition[1]

Thus if m^2 be the rational number optionally chosen, one shall have the identity :

$$N.1^2+(m^2-N)=m^2,$$

or

$$N.1^2-(N-m^2)=m^2$$

Then by applying Brahmagupta's Corollary we get

$$N(2m)^2+(m^2 \backsim N)^2=(m^2+N)^2$$

$$\therefore N\left(\frac{2m}{m^2 \backsim N}\right)+1=\left(\frac{m^2+N}{m^2 \backsim N}\right)^2$$

Hence

$$x=\frac{2m}{m \backsim N},\ y=\frac{m^2+N}{m^2 \backsim N}$$

where m is any rational number, is a solution of the equation $Nx^2+1=y^2$.

This rationol solution of the *varga-prakṛti* which was used by Śrīpati in 1039 A.D. was rediscovered in Europe by Brouncker in 1657.

We shall close this discussion by taking an illustration from Bhāskara II :

Problem : Tell me, O mathematician, what is that square which multiplied by 8 becomes, together with unity, a square; and what square multiplied by 11 and increased by unity, becomes a square.

This means that we have to solve the equations :

$$8x^2+1=y^2 \quad ......(i)$$

$$11x^2+1=y^2 \quad ......(ii)$$

In the second example, let us assume 1 as the lesser root. Following the method of Śrīpati, let us multiply its square by the *prakṛti* (here in eq. ii, prakṛti is 11), then let us subtract 2 (an optional number) and then extracting the square-roots we

1. Śrīpati, *Siddhānta-śekhara* XIV. 33

get the greater root as 3 Hence the statement for the com position is

$m=11 \quad l=1 \quad g=3 \quad i=-2$

$\qquad\qquad l=1 \quad g=3 \quad i=-2$

Here m=multiplier (*guṇaka* or *prakṛti*) l=lesser root (*kaniṣṭha mūla*) g=greater root (*jyeṣṭha mūla*) and i=interpolator (*kṣepa*)

Here we have set down successively the lesser root greater root and interpolator and below them again set down the same (See Brahmagupta s Lemmas described by Bhāskara II) Now proceeding as before we obtain the roots for the additive 4

$l=6 \quad g=20 \quad (\text{for}) \; i=4$

Then by the rule

> If the interpolator (of a *varga-prakṛti* or Square-nature) divided by the square of an optional number be the interpolator (of another Square-nature) then the two roots (of the former) divided by that optional number will be the roots (of the other) Or if the interpolator be multiplied their roots should be multiplied [1]

are found the roots for the additive unity

$l=3 \quad g=10 \quad (\text{for}) \; i=1$

Whence by the Principle of Composition of Equals we get the lesser and greater roots $l=60$ $g=199$ (*for*) $i=1$ In this way an infinite number of roots can be deduced

Alternative method Bhaskara II has given another method for finding the two roots for the additive unity

> Or divide twice an optional number by the difference between the square of that optional number and the *prakṛti* This (quotient) will be the lesser root (of a Square-nature) when unity is the additive From that (follows) the greater root [2]

1 इष्टवर्गहृतः क्षेपः क्षेपः स्यादिष्टभाजिते ।
मूले ते स्तोऽथवा क्षेपः क्षुण्णः क्षुण्णे तदा पदे ॥

Bijagaṇita II 5

2 *Siddhānta śekhara* XIV 32

Let us solve the first example $8x^2+1=y^2$. We assume the optional number to be 3. Its square is 9; the *prakṛti* of multiplier is 8, their difference is 9–8=1. Dividing by this twice the optional number (2×3, i e. 6), namely 6, we get the lesser root for the additive unity as 6. Whence proceeding as before, we get the greater to be 17. Thus here $x=6$ and $y=17$.

Let us use this method for the equation $11x^2+1=y^2$. Let the optional number be 3. Its square is 9. multiplier or *prakṛti* is 11; the difference is 11–9=2; dividing by this twice the optional number (2×3), namely 6, we get 6/2=3, which is the lesser root. Consequently the greater root would be 10. Thus for this equation $x=3$ and $y=10$.

Solution in Positive Integers

The Indian algebraists usually aimed at obtaining solutions of the *varga-prakṛti* or Square nature in positive integers or *abhinna*. The tentative methods of Brahmagupta and Śrīpati always did not furnish solutions in positive integers. These authors, however, discovered that if the interpolator of auxiliary equation in the tentative method be ±1, ±2 or ±4, an integral solution of the equation $Nx^2+1=y^2$ can always be found. Thus Śrīpati says:

> If 1, 2 or 4 be the additive or subtractive (of the auxiliary equation), the lesser and greater roots will be integral (*abhinna*)[1].

(i) If $k=\pm 1$, then the auxiliary equation will be

$$N\alpha^2 \pm 1 = \beta$$

where α and β are intergers. Then by Brahmagupta' Corollary we get

$$x = 2\alpha\beta \text{ and } y = \beta^2 + N\alpha^2$$

as the required first solution in positive integers of the equation $Nx^2+1=y^2$

1. इष्टवर्गप्रकृत्योर्विवरं तेन वा भजेत् ।
द्विघ्नमिष्टं कनिष्ठं तत् पदं स्यादेकसंयुतौ ।
ततो ज्येष्ठमिहानन्त्यं भावनातस्तथेष्टतः ॥

Bījagaṇita, Varga-Prakṛti, 5-6

(ii) Let $k=\pm 2$, then the auxiliary equation is

$$N\alpha^2 \pm 2 = \beta^2$$

By Brahmagupta's Corollary, we have

$$N(2\alpha\beta)^2 + 4 = (\beta^2 + N\alpha^2)^2$$

or $$N\,(\alpha\beta)^2 + 1 = \left(\frac{\beta^2 + N\alpha^2}{2}\right)^2$$

Hence the required first solution is

$$x = \alpha\beta \quad y = \tfrac{1}{2}(\beta^2 + N\alpha^2)$$

Since $$N\alpha^2 = \beta^2 \mp 2$$

we have $\frac{1}{2}\,(\beta^2 + N\alpha^2) = \beta^2 \mp 1 =$ a whole number

(iii) Now suppose $k = +4$, so that

$$N\alpha^2 + 4 = \beta^2$$

With an auxiliary equation like this the first integral solution of the equation $Nx^2 + 1 = y^2$ is

$$x = \tfrac{1}{2}\alpha\beta$$
$$y = \tfrac{1}{2}(\beta^2 - 2),$$

if α is even, or

$$x = \tfrac{1}{2}\alpha(\beta^2 - 1)$$
$$y = \tfrac{1}{2}\ \beta\ (\beta^2 - 2)$$

if β is odd

Thus we find Brahmagupta saying

> In the case of 4 as additive the square of the second root diminished by 3 then halved and multiplied by the second root will be the (required) second root the square of the second root diminished by unity and then divided by 2 and multiplied by the first root will be the (required) first root (for the additive unity)[1]

Datta and Singh has given the following *rationale* of this solution

$$\text{Since } N\,\alpha^2 + 4 = \beta^2 \qquad (i)$$

$$\text{we have } N\,(\alpha/2)^2 + 1 = (\beta/2)^2. \qquad (ii)$$

Then by Brahmagupta s Corollary we get

$$N\,(\alpha\beta/2)^2 + 1 = \left(\frac{\beta^2}{4} + N\frac{\alpha^2}{4}\right)^2$$

---

1 चतुरधिकेऽन्त्यपदकृतिस्त्र्यूनार्दलिताऽन्त्यपद गुणाऽन्त्यपदम् ॥
अन्त्यपद कृतिर्व्येकार्दिहताऽऽद्यपदाहताऽऽद्यपदम् ॥ —*BrSpSi* XVIII 67

Substituting the value of N in the right-hand side expression from (i), we have

$$N\left(\frac{\alpha\beta}{2}\right)^2+1=\left(\frac{\beta^2-2}{2}\right)^2 \qquad \text{(iii)}$$

Composing (ii) and (iii),

$$N\left\{\frac{\alpha}{2}(\beta^2-1)\right\}^2+1=\left\{\frac{\beta}{2}(\beta^2-3)\right\}^2$$

Hence $x=\frac{1}{2}\alpha\beta$, $y=\frac{1}{2}(\beta^2-2)$;
and $x=\frac{1}{2}\alpha(\beta^2-1)$, $y=\frac{1}{2}\beta(\beta^2-3)$;
are solutions of $Nx^2+1=y^2$.

If β be even, the first values of (x,y) are integral. If β be odd, the second values are integral.

(iv) Finally, suppose $k=-4$; the auxiliary equation is

$$N\alpha^2-4=\beta^2$$

Then the required first solution in positive integers of
$Nx^2+1=y^2$ is
$x=\frac{1}{2}\alpha\beta(\beta^2+3)(\beta^2+1)$
$y=(\beta^2+2)\{\frac{1}{2}(\beta^2+3)(\beta^2+1)-1\}$.

Brahmagupta says:

> In the case of 4 as subtractive, the square of the second is increased by three and by unity; half the product of these sums and that as diminished by unity (are obtained). The latter multiplied by the first sum less unity is the (required) second root; the former multiplied by the product of the (old) roots will be the first root corresponding to the (new) second root.[1]

The rationale of this solution, as given by Datta and Singh is as follows:

$$N\alpha^2-4=\beta^2 \qquad (1)$$
$$N(\alpha/2)^2-1=(\beta/2)^2$$

Hence by Brahmagupta's Corollary, we get

$$N\left(\frac{\alpha\beta}{2}\right)^2+1=\left(\frac{\beta^2}{4}+N\frac{\alpha^2}{4}\right)^2$$

1. चतुरूनेऽन्त्यपदस्य कृतिः त्र्येकयुते वधदलं पृथक्सैकम् ।
सैकपदघ्नमन्त्यं पदवधगुणमाद्यमन्त्यपदम् ॥ *BrSpSi. XVIII. 68*

$$=\{\tfrac{1}{2}(\beta^2+2)\}^2 \qquad (ii)$$

Again applying the Corollary, we get

$$N\{\tfrac{1}{2}\alpha\beta(\beta^2+2)\}^2+1=\{\tfrac{1}{2}(\beta^4+4\beta^2+2)\}^2 \qquad (iii)$$

Now by *the Lemma we obtain from (ii) and (iii)*

$$N\{\tfrac{1}{2}\alpha\beta(\beta^2+3)(\beta^2+1)\}^2+1$$
$$=[(\beta^2+2)\{\tfrac{1}{2}(\beta^2+3)(\beta^2+1)-1\}]^2$$

Hence $x=\tfrac{1}{2}\alpha\beta(\beta^2+3)(\beta^2+1)$

$y=(\beta^2+2)\{\tfrac{1}{2}(\beta^2+3)(\beta^2+1)-1\}$

is a solution of $Nx^2+1=y^2$

This can be proved without difficulty that these values of x and y are integral Since if β is even β^2+2 is also even And hence the above values of x and y are integral On the other hand, if β is odd, β^2 is also odd under these conditions β^2+1 and β^2+3 are even In this also, therefore the above values must be integral

Putting $p=\alpha\beta$ $q=\beta^2+2$, *we can write the above solution in the form*

$$x=\tfrac{1}{2}p(q^2-1)$$
$$y=\tfrac{1}{2}q(q^2-3)$$

This was the form in which the solution was found by Euler

Cakravala or Cyclic Method

We have shown in the preceding articles that the most fundamental step in Brahmagupta's method for the general solution in positive integers of the equation

$$Nx^2+1=y^2$$

where N is a non-square integer, is *to form an auxiliary equation* of the kind

$$Na^2+k=b^2$$

where a and b are positive integers and $k=\pm1$, ±2 or ±4 From this auxiliary equation by the Principle of Composition applied repeatedly whenever necessary one can derive, as we have already shown above, one positive integral solution of the original *Varga prakrti* or Square-Nature. And thence again by means of the same principle an infinite number of other solutions in integers can be obtained How to form an auxiliary equation of

this type was a problem write Datta and Singh which could not be solved completely nor satisfactorily by Brahmagupta In fact Brahmagupta had to depend on trial Success in this direction was, however remarkably attained by Bhāskara II He evolved a simple and elegant method which assisted in deriving an auxiliary equation having the required interpolator $\pm 1, \pm 2$ or ± 4 simultaneously with its two integral roots from another auxiliary equation empirically formed with any simple integral value of the interpolator positive or negative This method has been technically known as *Cakravāla* or the *cyclic method* This is so called because it proceeds as in a circle, the same set of operations being applied again and again in a continuous round For the details of this method our reader is requested to consult the *Algebra of Bhāskara II and the narrative on this method as* given by Datta and Singh under the title 'Cyclic Method in their *History of Hindu Mathematics Algebra* 1962 Edition, pp 161 72.

Solution of Indeterminate Quadratic Equation

It is remarkable to see that Brahmagupta was the first algebraist in the history of mathematics to find a general solution of the indeterminate quadratic equation

$$Nx^2 \pm c = y^2$$

in positive integers We have the following verse in the *Brāhmasphuṭasiddhānta* in this connection

> From two roots (of a Square nature or *varga-prakṛti*) with any given additive or subtractive by making (combination) with the roots for the additive unity other first and second roots (of the equation having) the given additive or subtractive (can be found)[1]

Let us take the following two equations.

$a_1k = an + b$ and $b_1k = bn + Na$

From them we get by eliminating n

$$a_1b - ab_1 = 1$$

1 रूप प्रक्षेपपदे पृथगिष्टक्षेप्यशोध्यमूलाभ्याम् ।
कृत्वाऽन्याद्यपदे ये प्रक्षेपे शोधने वेष्टे ॥ —*BrSpS*, XVIII 66

Hence $b_1 = \frac{a_1 b - 1}{a}$ = a whole number.

Now $n^2 - N = \frac{(a_1 k - b)^2 - Na^2}{a^2}$

$$= \frac{a_1^2 k^2 - 2bka_1 + k}{a^2}$$

$$= \frac{k(a_1^2 k - 2ba_1 + 1)}{a^2}$$

Therefore $\frac{k}{a^2}(a_1^2 k - 2ba_1 + 1)$ is a whole number

Since a k have no common factor, it follows that

$$\frac{a_1^2 k - 2ba_1 + 1}{a^2} = \frac{n^2 - N}{k} = k_1 = \text{an integer.}$$

Also $k_1 = \frac{n^2 - N}{k} = \frac{a_1^2 k - 2ba_1 + 1}{a^2}$

$$= \frac{a_1^2(b^2 - Na^2) - 2ba_1 + 1}{a^2}$$

$$= \left(\frac{a_1 b - 1}{a}\right)^2 - Na_1 .$$

Thus having known a single solution in positive integers of the equation $Nx^2 \pm c = y^2$, says, Brahmagupta, an infinite number of other integral solutions can be obtained by making use of the integral solutions of $Nx^2 + 1 = y^2$ If $(p\, q)$ be a solution of the former equation found empirically and if $(\alpha\, \beta)$ be an integral solution of the latter, then by the principle of Composition

$$x = p\beta \pm q\alpha \quad y = q\beta \pm Np\alpha$$

will be a solution of the former. Repeating the operations, we can easily deduce as many solutions as we like

FORM $Mn^2x^2 \pm c = y^2$

In this connection Brahmagupta says ·

If the remainder is that divided by a square, the first root is that divided by its root[1]

This seems to mean that if we have the equation

$Mn^2x^2 \pm c = y^2$ (1)

such that the multiplier (i.e. the coefficient of x^2) is divisible

1 वर्गच्छिन्ने गुणके प्रथम तन्मूल भाजित भवति । —BrSpSi XVIII 70

by n^2, then we are justified in saying that if we put $nx=u$, the equation (i) becomes $Mu^2 \pm c = y^2$ (ii),
and clearly the first root of (i) is equal to the first root of (ii) divided by n. The corresponding second root will be the same for both the equations.

FORM $a^2x^2 \pm c = y^2$:

We find Brahmagupta giving the following rule in this connection: This is a solution of a particular form of a *varga-prakṛti* or Square-nature.

> If the multiplier be a square, the interpolator divided by an optional number and then increased and decreased by it, is halved. The former (of these results) is the second root; and the other divided by the square-root of the multiplier is the first root.[1]

Thus the solutions of the equation

$$a^2x^2 \pm c = y^2$$

are:

$$x = \frac{1}{2a}\left(\frac{\pm c}{m} - m\right)$$

$$y = \frac{1}{2}\left(\frac{\pm c}{m} + m\right)$$

where m is an arbitrary number.

Bhāskara II and Nārāyana have also given the same solutions as proposed by Brahmagupta.

Rational Geometrical Figures

In the days of the *Taittirīya Saṁhitā* and the *Śatapatha Brāhmaṇa*, Indian mathematicians got familiarity with the solution of such equations

$$x^2 + y^2 = z^2$$

and the results were arrived geometrically on the basis of the law of rectangle as propounded by Baudhāyana in the Śulba Sūtras and which goes by his name. The reader is referred to the Chapter on Baudhāyana, the first Geometer in the author's "*Founders of Sciences in Ancient India*". Baudhāyana (c 800 B.C.) gave a

1. वर्गे गुणके क्षेपः केनचिदुद्धृतयुतोनितो दलितः ।
प्रथमोऽन्यमूलमन्यो गुणकारपदोद्धृतः प्रथमः ॥ —*BrSpSi*. XVIII. 69

method of transforming a rectangle into a square, which is equivalent to the algebraic identity:

$$mn = \left(m - \frac{m-n}{2}\right) - \left(\frac{m-n}{2}\right)$$

where m, n, are any two arbitrary numbers.

Brahmagupta in connection with the solution of rational triangles says

> The square of the optional (*iṣṭa*) side is divided and then diminished by an optional number; half the result is the upright, and that increased by the optional number gives the hypotenuse of a rectangle.[1]

We shall put these statements of Brahmagupta in the algebraic language thus: If m, n be any two rational numbers, then the sides of a right-angled triangle will be

$$m, \ \frac{1}{2}\left(\frac{m^2}{n} - n\right), \ \frac{1}{2}\left(\frac{m^2}{n} + n\right)$$

This Sanskrit term *iṣṭa* may either mean "given" or "optional". With the former meaning the rule would imply the method of finding rational right angles having a given leg.

Brahmagupta was the first to give a solution of the equation $x^2+y^2=z^2$ in integers. His solution is

$$m^2-n^2, 2mn, m^2+n^2$$

m and n being two unequal integers.

Thus if $m=7$ and $n=4$ then $m^2-n^2=33$, $2mn=56$ and $m^2+n^2=65$, then the three numbers 33, 56 and 65 bear the relation $33^2+56^2=65^2$.

Mahavira (850 A.D.) also states

> The difference of the squares (of two elements) is the upright, twice their product is the base and the sum of their squares is the diagonal of a generated rectangle.[2]

Isosceles Triangles with Integral Sides. The following statement of Brahmagupta in this connection is very significant

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1 इष्टस्य भुजस्य कृतिर्भक्ता भेद्येन सदलं कोटिः ।
आयतचतुरस्रस्य च भेद्येनाधिका कर्णो ॥ —*BrSpSi* XII 33

2 GSS VII 90½

> The sum of the squares of two unequal numbers is the side; their product multiplied by two is the altitude. and twice the difference of the squares of those two unequal numbers is the base of an isosceles triangle.[1]

Thus if m, n be two integers such that m is not equal to n, the sides of all rational isosceles triangles with integral sides are given by

$$m^2+n^2,\ m^2+n^2,\ 2(m^2-n^2)$$

and the altitude of the triangle is $2mn$.

This method was also followed by Mahāvīra and other Indian mathematicians. In fact, their solutions are based on the juxtaposition of two rational right triangles, equal so that they have a common leg. It is remarkably a powerful device, for every rational triangle or quadrilateral may be formed by the juxtaposition of two or four rational right triangles.

Isosceles Triangles with a Given Altitude

Here we have a rule given by Brahmagupta for finding out all rational isosceles triangles possessing the same altitude :

> The (given) altitude is the producer (*karaṇi*). Its square divided by an optional number is increased and diminished by that optional number. The smaller is the base and half the greater is the side.[2]

Thus if m be any rational number then for a given definite altitude a, the sides of the rational isosceles triangles are $\frac{1}{2}\left(\frac{a^2}{m}+m\right)$ each and the base is $\frac{a^2-m^2}{m}$. We shall illustrate it by an example taken from the commentary of Pṛthūdaka Svāmi The given altitude is 8; let us take any rational number $m=4$ then the two equal sides of the isosceles are given by $\frac{1}{2}\left(\frac{8^2+4^2}{4}\right)=$ 10 each and the base is $\frac{8^2-4^2}{4}=12$. Thus the three sides of the

1. कृति युतिर् सदृशराश्योर्बाहुर्घातो द्विसंगुणो लम्बः ।
कृत्यन्तरमसदृशयोर्द्विगुणं द्विसमत्रिभुज भूमिः ॥ —*BrSpSi*. XII. 33

2. करणी लम्बस्तत्कृतिरिष्टहृतेष्टेन संयुताऽल्पा भूः ।
अधिको द्विहृतो बाहुः संज्ञेयो यत्नथो वर्गः ॥ —*BrSpSi*. XII. 37.

rational isosceles triangle with altitude 8 are (10 10, 12)

Rational Scalene Triangles Brahmagupta lays down the following rule in the case of rational scalene triangle,

> The square of an optional number is divided twice by two arbitrary numbers the moieties of the sums of the quotients and (respective) optional numbers are the sides of a scalene triangle the sum of the moities of the differences is the base [1]

In other words if m p q are any rational numbers then the sides of a rational scalene triangle are

$$\frac{1}{2}\left(\frac{m^2}{p}+p\right),\ \frac{1}{2}\left(\frac{m^2}{q}+q\right).$$

$$\frac{1}{2}\left(\frac{m^2}{p}-p\right)+\frac{1}{2}\left(\frac{m^2}{q}-q\right)$$

Here the altitude (m) area and segments of the base of this triangle are all rational

Thus putting $m=12$ $p=6$, and $q=8$ in Brahmagupta s general equation Prthūdaka Svāmi derives a scalene triangle with sides (13 15) and (14) altitude (12) area (84 and the segments of the base (5) which are all integral numbers

$$\frac{1}{2}\left(\frac{m^2}{p}+p\right)=\frac{1}{2}\left(\frac{12^2}{6}+6\right)=15$$

$$\frac{1}{2}\left(\frac{m^2}{q}+q\right)=\frac{1}{2}\left(\frac{12^2}{8}+8\right)=15$$

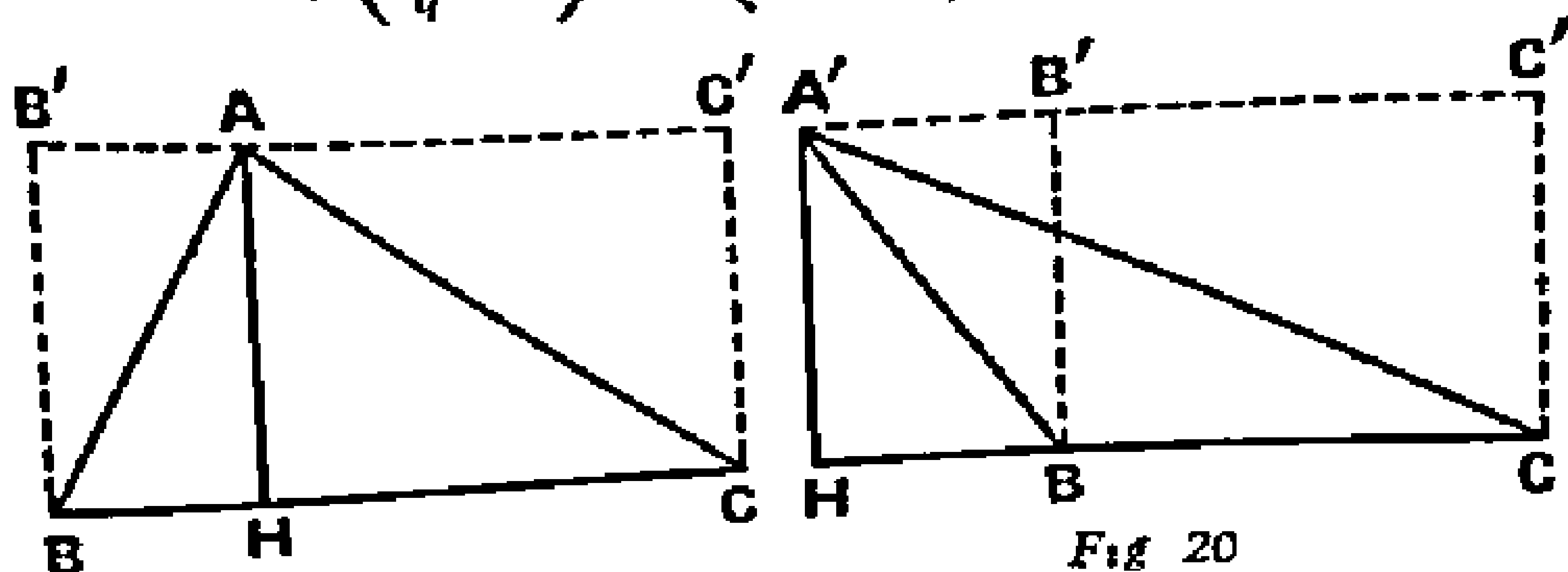


Fig 19

Fig 20

1 इष्टद्वयेन भक्तो द्विधेष्ट वर्गो फलेष्टयोगार्धे ।
विषमत्रिभुजस्य भुजाविष्टोनफलार्धयोगो भूः ॥ —*BrSpSi* XII 34

Thus the two sides of the rational scalene triangle are 15 and 13 The base is

$$\tfrac{1}{2}\left(\frac{12^2}{6}-6\right)+\tfrac{1}{2}\left(\frac{12^2}{8}-8\right)=9+5=14$$

The altitude is $m=12$ area is equal to $\dfrac{\text{base}\times\text{altitude}}{2}$

$=\dfrac{14\times 12}{2}=$ and the segments are $\sqrt{13^2-12^2}=5$ and $\sqrt{(15^2-12^2)}=9$ Thus they are all integers

Rational Isosceles Trapeziums

Brahmagupta has given us a method of obtaining such isosceles trapeziums whose sides diagonals, altitude segments and area are all rational numbers His rule is as follows

The diagonals of the rectangle (generated) are the flank sides of an isosceles trapezium the square of its side is divided by an optional number and then lessened by that optional number and divided by two (the result) increased by the upright is the base and lessened by it is the face [1]

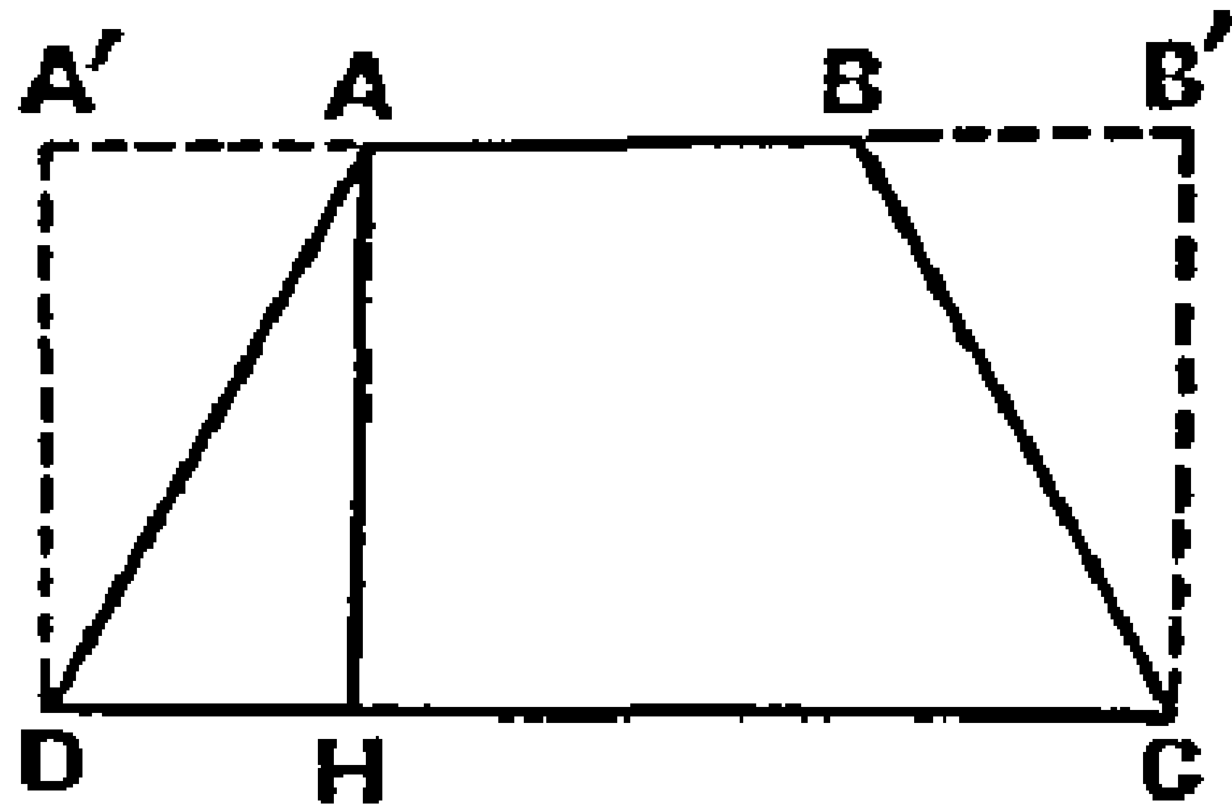


Fig 21

Here in the figure we have the isosceles trapezium ABCD of which C D is the base and A B is known as the fase According to Brahmagupta s rule, we have (p being the optional number)

$$CD = \tfrac{1}{2}\left(\frac{4m^2n^2-p}{p}\right)+\left(m^2-n^2\right) \qquad \text{(base)}$$

$$AB = \tfrac{1}{2}\left[\frac{4m^2n^2}{p}-p\right]-\left[m^2-n^2\right] \qquad \text{(face)}$$

$$DH - (m^2-n^2) \qquad \text{(upright)}$$

1 आयतकर्णौ बाहू भुजकृतिरिष्टेन भाजितेष्टोना ।
द्विहृता कोट्यधिका भूर्मुखमूना द्विसमचतुरस्रे ॥ —*BrSpSi* XII 36

$AD = BC = m^2+n^2$ (the sides of the trapezium)

HC = base − upright = $\frac{1}{2}\left[\frac{4m^2n^2}{p} - p\right]$ (segment)

$AC = BD = \left[\frac{4m^2n^2}{p} + p\right]$ (diagonal)

$AH = 2\,mn$ (altitude)

$ABCD = mn\left[\frac{4m^2n^2}{p} - p\right]$ (area)

By chosing the values of m n and p suitably, the values of all the dimensions of the isosceles trapezium can be made integral Prthūdaka Svāmī starts with the rectangle (5, 12, 13) and suitably takes p as 6, then he calculates out the dimensions of the trapezium: flank sides (AD and BC) = 13, base = 14, and base = 4 altitude (AH) = 12 segments of base (DH and HC) = 5 and 9, diagonals (AC and BD) = 15, area ABCD = 108. All these values are integers

In this example, the rectangle chosen is (5, 12, 13) which is AA' DH, where $AD = m^2 + n^2 = 13$

and $DH = m^2 - n^2 = 5$

whence by adding the two we have

$2m^2 = 18$

This gives the value of $m = 3$, and hence $n = 2$. Prthūdaka Svāmī has taken the value of $p = 6$ by choice. Putting these values of m, n and p, the values for the dimensions of the isosceles trapezium follow from the expressions given by Brahmagupta

$$CD = \frac{1}{2}\left(\frac{4\,3^2\,2^2}{6} - 6\right) + \left(3^2 - 2^2\right) = 9+5 = 14 \text{ (base)}$$

Face = $9-5=4$

Sides $AD = BC = 3^2+2^2 = 13$

and so on for the other dimensions

Rational Trapeziums With Three Equal Sides

This problem is very much the same as one of the rational isosceles tpapezium with the only difference that in this case one of the parallel sides is also equal to the slant sides. We

have the following solution of this problem from Brahmagupta

> The square of the diagonal (of a generated rectangle) gives three equal sides the fourth (is obtained) by subtracting the square of the upright from thrice the square of the side (of that rectangle) If greater it is the base if less it is the face [1]

As before the rectangle generated from m n is given by (m^2-n^2 $2mn$ m^2+n^2) that is these are the three sides of the right triangle which correspond to the two sides and the diagonal of the rectangle generated by them Let us suppose we have a trapezium ABCD whose sides AB BC and AD are equal then

$$AB = BC = AD = (m^2+n^2)^2$$

$$CD = 3(2mn)^2 - (m^2-n^2)^2 = 14\,m^2n^2 - m^4 - n^4$$

$$\text{or } CD = 3(m^2-n^2)^2 - (2mn)^2 = 3m^4+3n^4 - 10\,m^2n^2$$

Prthūdaka Svāmi has taken an illustration where $m=2$ $n=1$ and he deduces two rational trapeziums with three equal sides (25 25 25 39) and (25 25 25 11)

The segment (CH) altitude (AH) diagonals (AC, BD) and area of this trapezium are also rational and given by

$$CH \text{ (segment)} = 6\,m^2n^2 - m^4 - n^4$$

$$AH \text{ (altitude)} = 4\,mn\,(m^2-n^2)$$

$$AC = BD \text{ (diagonals)} = 4\,mn\,(m^2+n^2)$$

$$ABCD \text{ (area)} = 32m^3n^3\,(m^2-n^2)$$

Rational Inscribed Quadrilaterals

We find in the *Brāhmasphuṭasiddhānta* a remarkable proposition formulated by Brahmagupta

> To find all quadrilaterals which will be inscribable within circles and whose sides diagonals perpendiculars, segments (of sides and diagonals by perpendiculars from vertices as also of diagonals by their intersection) areas and also the diameters of the

1 कर्णकृतिस्त्रिसम भुजास्त्रयश्चतुर्थो विशोध्य कोटि कृतिम् ।
बाहुकृतेस्त्रिगुणाय यद्यधिको भूर्मुखं हीन ॥ —*BrSpSi* XII 37

circumscribed circles will be expressible in integers Such quadrilaterals we shall call as *Brahmagupta Quadrilaterals*

The solution of this formidable problem has been given by Brahmagupta as follows.

> The upright and bases of two right angled triangles being reciprocally multiplied by the diagonals of the other will give the sides of a quadrilateral of unequal sides: (of these) the greatest is the base, the least is the face, and the other two sides are the two flanks [1]

Taking Brahmagupta's integral solution, the sides of the two right triangles of reference are given by

(i) m^2-n^2, $2mn$, m^2+n^2,

(ii) p^2-q^2, $2pq$, p^2+q^2,

where m, n, p, q are integers Then the sides of the *Brahmagupta Quadrilateral* are

$$(m^2-n^2)(p^2+q^2),\ (p^2-q^2)(m^2+n^2),$$
$$2mn(p^2+q^2),\ 2pq(m^2+n^2) \qquad \text{(Arrangement A)}$$

Pṛthūdaka Svāmi has illustrated the rational inscribed quadrilateral by taking an example of the right angle triangles

(i) (3,4,5) ($m^2-n^2=3$, $m^2+n^2=5$, whence $m=2, n=1$)

(ii) (5,12,13) ($p^2-q^2=5$, $p^2+q^2=13$, whence $p=3, q=2$)

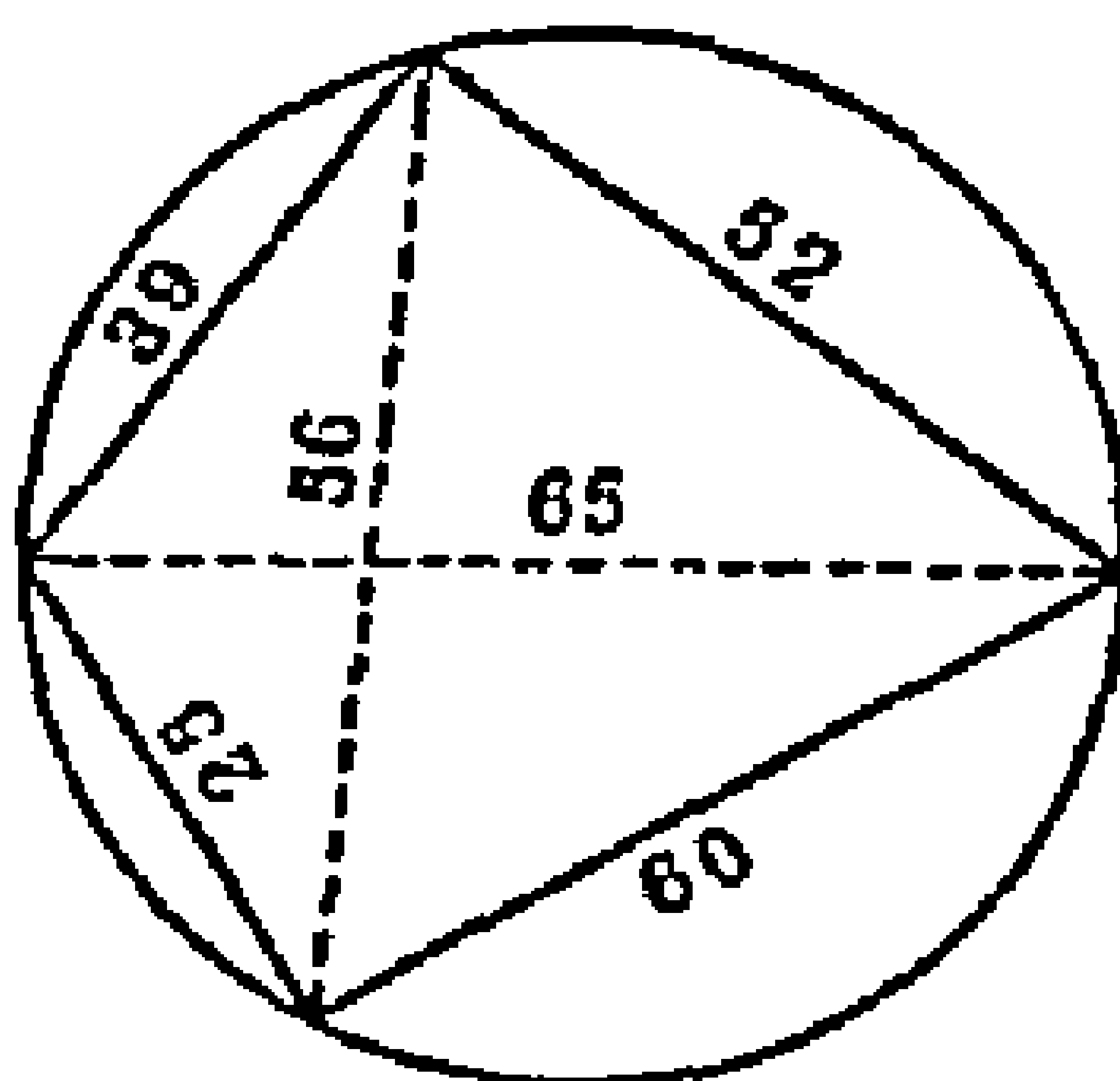


Fig 22

Substituting these values in the above equations, we get the sides of the quadrilateral as (39 25, 52 and 60) [2]

1 जात्यद्वयकोटिभुजा परकर्णगुणा भुजाश्चतुर्विषमे ।
अधिको भूर्मुखमल्पो बाहुद्वितयं भुजावन्यौ ॥ —*BrSpSi*. XII 38

2 The diagonals of this quadrilateral are given by Bhāskara II as 56 (=3,12+4,5) and 63 (=4,12+3,5)

(*Cont. on page 268*)

Mahāvira (*GSS* VII 103½) has given the details for finding out the diagonals, altitudes, segments and areas of such inscribed rational quadrilaterals Bhāskara II has given an example of such cyclic quadrilaterals with sides (68 51 40, 75), which was obtained by taking right triangles (3, 4 5) and (15 8, 17) the diagonals are 77 and 85 altitude is 308/5, segments are 144/5 and 231/5 and area is 3234 (*Lilavati*)

Chasles[2] and Kumar[3] have demonstrated the great significance of the results of Brahmagupta In fact, as said by Datta and Singh according to the *sequence* in which the quantities relating to the dimensions of sides are taken, there will be two varieties of the Brahmagupta quadrilaterals, (i) in one the two diagonals intersect at right angles (ii) and in the other they intersect obliquely. The arrangement A given previously gives the quadrilateral of the first kind i e the diagonals intersecting at right angles The following arrangement B will give the quadrilateral of the second kind with diagonals intersecting obliquely

$$(p^2-q^2)(m^2+n^2) \quad (m^2-n^2)(p^2+q^2)$$

$$2mn(p^2+q^2) \quad 2pq(m^2+n^2) \qquad \text{(Arrangement B)}$$

Alternatively the following arrangement also gives the rational cyclic quadrilateral with diagonals intersecting obliquely

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(*Cont from page 267*)

In fact Bhāskara II has given the following expressions for the three varieties of the Brahmagupta Quadrilaterals

For sequences of sides in arrangement A

$2pq(m^2-n^2)+2mn(p^2 \quad q^2) \quad 4mnpq+(p^2 \quad q^2)(m^2-n^2)$

For sequences of sides in arrangement B

$2pq\ (m^2 \quad n^2)+2mn\ (p^2-q_2) \quad (p_2+q^2)(m^2+n^2)$

For sequences of sides in arrangement C

$4mnpq+(p^2 \quad q^2)(m^2-n^2) \quad (p^2+q^2)(m^2+n^2)$

The diameter of the circumscribed circle in every case is given by $(p^2+q^2)(m^2+n^2)$

2 M Chasles *Apercu historique sur l origine et developpment des methodes en geometrie* Paris 1875 pp 436 ff

3 E,E, Kummer *Über die Vierecke deren Se ten und Diogonalen rational sind Journ für Math* XXXVII 1848 pp 1 20

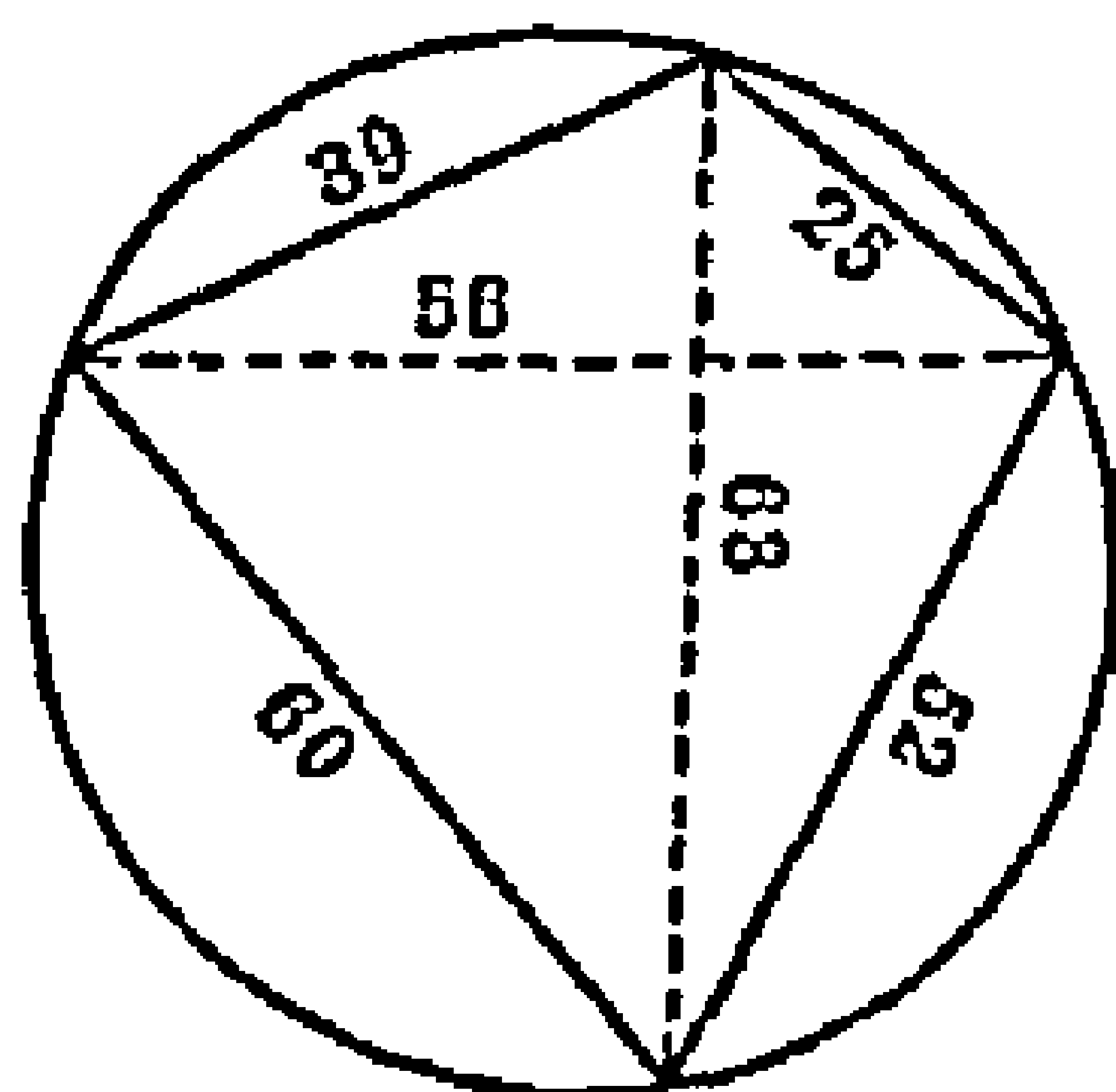


Fig 23

$(p^2-q^2)(m^2+n^2)\quad 2mn(p^2+q^2)$

$(m^2-n^2)(p^2+q^2)\quad 2pq(m^2+n^2)$

(Arrangement C)

Linear Functions Made Squares or Cubes

Varga-Kuṭṭaka It is the name given to the indeterminate equation of the type

$$bx+c=y^2$$

The term may be translated into English as 'Square pulveriser' We may rewrite the equation in the form

$$x=\frac{y^2-c}{b}$$

When expressed in this form, we have a problem of finding a square (*varga*) which being diminished by c will be exactly divisible by b, which closely reminds the problem solved by the *kuṭṭaka* method (method of pulveriser)

To solve a problem of this type with the result appearing as integers, Indian algebraists used to assume suitable arbitrary values for y and then to solve the equation for x We find Brahmagupta stating as follows in this connection

> The residue of the Sun on Thursday is lessened and then multiplied by 5 or by 10. Making this (result) an exact square, within a year, a person becomes a mathematician [1]
>
> The residue of any optional revolution lessened by 92 and then multiplied by 83 becomes together with unity a square A person solving this within a year is a *mathematician* [2]

Put in other words, this means that one has to solve the following equations

(i) $5x-25 = y^2$

(ii) $10x-100 = y^2$

(iii) $83x-7635 = y^2$

Pṛthūdaka Svāmi the commentator on the *Brāhmasphuṭa-siddhānta* proceeds to solve these equations as follows :

(1 1) Suppose $y = 10$, then $x = 125$. Or put $y = 5$, then $x = 10$

(2 1) Suppose $y = 10$ then $x = 20$

(3 1) Assume $y = 1$, then $x = 92$

He then remarks that by virtue of the multiplicity of suppositions there will be an infinitude of solutions in every case. But no method has been given either by Brahmagupta or his commentator to obtain the general solution.

Double Equations of the First Degree

Perhaps we have the earliest reference of the simultaneous indeterminate quadratic equations of the type

$$x \pm a = u^2$$
$$x \pm b = v^2$$

in the *Bhakaśālī Manuscript* (Folio 59, recto)

Brahmagupta gives the solution of such simultaneous indeterminate quadratic equations of a general case as follows

> The difference of the two numbers by the addition or subtration of which another number becomes a square is divided by an optional number and then increased or decreased by it. The square of half the result diminished or increased by the greater or smaller (of the given number) is the number (required)[1]

Expressed in the language of algebra shall have

$$= \left\{ \frac{1}{2} \left(\frac{a-b}{m} \pm m \right) \right\}^2 \mp a$$

1 याभ्या कृरिधिको नस्नदन्तर हृन युनो न मिष्टेन ।
तद्दल कृतिरधिकोनाऽधिकयो रधिको न यो राशि ॥ —*BrSpSi* XVIII 74

$$\text{or } x=\left\{\tfrac{1}{2}\left(\frac{a-b}{m}\mp m\right)\right\}^2\mp b$$

where m is an arbitrary number.

Datta and Singh has given the *rationale* of this method as follows:

$$u^2=x\pm a;\ v^2=x\pm b.$$

From them, we have $u^2-v^2=\pm a\mp b$

Therefore $u-v=m$

$$\text{and } u+v=\frac{\pm a\mp b}{m},$$

where m is arbitraty. Hence

$$u=\tfrac{1}{2}\left(\frac{\pm a\mp b}{m}+m\right)=\pm\tfrac{1}{2}\left(\frac{a-b}{m}\pm m\right)$$

Since it is obviously immaterial whether u is taken as positive or negative, we have

$$u=\tfrac{1}{2}\left(\frac{a-b}{m}\pm m\right)$$

Similarly $v=\tfrac{1}{2}\left(\frac{a-b}{m}\mp m\right)$

$$\text{Therefore } x=\left\{\tfrac{1}{2}\left(\frac{a-b}{m}\pm m\right)\right\}^2\mp a.$$

$$\text{or } x=\left\{\tfrac{1}{2}\left(\frac{a-b}{m}\mp m\right)\right\}^2\mp b.$$

where m is an arbitrary number.

Now we shall take up another particular case, for which Brahmagupta has given a rule:

> The sum of the two numbers the addition and subtraction of which make another number (severally) a square, is divided by an optional number and then diminished by that optional number. The square of half the remainder increased by the subtractive number is the number (required)[1].

In the algebraic notations we shall express it as follows:

—BrSpSi XVIII, 71

$$x = \left\{ \tfrac{1}{2} \left\{ \frac{a+b}{m} - m \right\} \right\}^2 + b.$$

Further, Brahmagupta has at one place given a solution of the following double equations of the first degree :

$$\left. \begin{array}{l} ax+1=u^2 \\ bx+1=v^2 \end{array} \right\}$$

in the following words :

> The sum of the multipliers multiplied by 8 and divided by the square of the difference of the multipliers is the (unknown) number. Thrice the two multipliers increased by the alternate multiplier and divided by their difference will be the two roots.[1]

The solutions indicated in the above statement are :

$$x = \frac{8(a+b)}{(a-b)^2},\ u = \frac{3a+b}{a-b},\ v = \frac{a+3b}{a-b}$$

Multiple Equations

We shall cite here an interesting elegant problem in which three or more functions, linear or quadratic, of the unknowns have to be made squares or cubes. An astronomical problem of this type has been set by Bhāskara in the *Laghu-Bhāskarīya*[2].

> The sum, the difference and the product increased by 1, of the residues of the revolution of Saturn and Mars—each is perfect square.

We shall frame this problem like this:

> To find two numbers x and y such that the expressions $x+y$, $x-y$, and $xy+1$ are each a perfect square.

Brahmagupta gives the following solution to such a problem

> A square is increased and diminished by another. The sum of the result is divided by the square of half

---

1 गुणकयुतिरष्टगुणिता गुणकान्तर वर्ग भाजिता राशि. ।
गुणकौ त्रिगुणौ व्यस्ताधिकौ हृतावन्तरेण पदे ॥ *BrSpSi* XVIII 71

2 शेषौ मण्डलजौ यमक्षितिजयोः संयुक्तविश्लेषितौ ।—
अन्योन्याहतविग्रहौ च पददौ रूपेण संयोजितौ ।
एवं साधु विचिन्त्य वर्गविधिना त्रिचक्रमाद् वत्सरैः ।
संगण्या युगणार्कजक्षिति सुता कालेन कालोद्भवाः ॥ —*LBh*. VIII. 17

> their difference Those results multiplied (severally) by this quotient give the numbers whose sum and difference are squares as also their product together with unity [1]

Thus the solution is :

$$x = P(m^2+n^2)$$
$$y = P(m^2-n^2),$$

where $P = \dfrac{(m^2+n^2) + (m^2-n^2)}{[\frac{1}{2}\{(m^2+n^2) - (m^2-n^2)\}]^2}$

m, n being any rational numbers

Work on such problems has been considerably extended by Bhāskara II (1150 A D) and Nārāyana (1357 A D)

Solution of $axy = bx+cy+d$

The quadratic indeterminate equation of this type probably has been tried for the first time in the *Bakhaśāli Manuscript* Brahmagupta mentions the following method for such a quadratic equation The solution given by him is not his own He has taken it from an *unknown* author, whose name is not mentioned anywhere Brahmagupta quotes the solution as follows :

> The product of the coefficient of the factum and the absolute number together with the product of the coefficients of the unknowns is divided by an optional number Of the optional number and the quotient obtained the greater is added to the lesser (of the coefficients) and (the sums) are divided by the coefficient of the factum (The results will be the values of the unknowns) in the reverse order [2]

As has been indicated by Brahmagupta's commentator, Pṛthūdaka Svāmī this rule is to be applied to an equation containing the factum after it has been prepared by transposing the factum term to one side and the absolute term together with the simple unknown terms to the other, Then the solutions will be, m being an arbitrary rational number,

$$x = \frac{1}{a}(m+c)$$

1 वर्गोऽन्यकृतिगुणोनस्तद्योगान्तरार्धकृतिभक्तः ।
तद्गुणितौ युतिवियुतौ वर्गौ घाते च रूपयुते ॥ BrSpSi. XVIII. 72

2 भावितक रूपगुणना साम्यं वरेष्टभाजितेष्टाप्तयोः ।
महतोऽल्पकेऽधिकेऽल्पे वेयो भावितहृतो व्यस्तम् ॥ BrSpSi XVIII 60

$$y = \frac{1}{a}\left(\frac{ad+bc}{m}+b\right)$$

if $b>c$ and $m > \frac{ad+bc}{m}$ If these conditions be reversed then x and y will have their values interchanged.

Datta and Singh have given the following *rationale* of these solutions :

$$axy=bx+cy+d,$$

or $a^2xy-abx-acy=ad,$

or $(ax-c)\ (ay-b)=ad+bc$

Suppose $ax-c=m$, a rational number;

then $ay-b=\frac{ad+bc}{m}$.

Therefore

$$x= \frac{1}{a}(m+c)$$

$$y= \frac{1}{a}\left(\frac{ad+bc}{m}+b\right)$$

Or, we may put $ay-b=m$,

in that case, we shall have $ax-c=\frac{ad+bc}{m}$:

whence $x=\frac{1}{a}\left(\frac{ad+bc}{m}+c\right)$.

$$y= \frac{1}{a}(m+b)$$

Brahmagupta's own rule.

Whilst the rule given above is ascribed to an unknown author, Brahmagupta's own rule for the solution of a quadratic indeterminate equation involving a factum is as follows :

> With the exception of an optional unknown, assume arbitrary values for the rest of the unknowns, the product of which forms the factum The sum of the product of these (assumed values) and the (respective) coefficients of the unknowns will be absolute quantities. The continued products of the assumed values and of the coefficient of the factum will be the coefficient of the optionally (left out) unknown Thus the solution

> *is effected without forming an equation of the factum Why then was it done so ?*[1]

Datta and Singh think that the reference in the latter portion of this rule is to the method of the unknown author

"*Kim krtam tadatah ?* The principle underlying Brahmagupta s method is to reduce like the Greek Diophantus (c.275 A D) the given indeterminate equation to a simple determinate one by assuming arbitrary values for all the unknowns except one So undoubtedly it is inferior to the earlier method

We now take an illustrative example from Brahmagupta

> *On subtracting from the product of signs and degrees of the Sun three and four times (respectively) those quantities ninety is obtained Determining the Sun within a year (one can pass as a proficient) mathematician*

If we presume x to denote the signs and y the degrees of the Sun then the equation would be

$$xy-3x-4y=90$$

Prthudaka Svami solves it in two ways

(i) *Let us assume the arbitrary number to be 17 then*

$$x=\frac{1}{1}\left(\frac{90\;1+3\;4}{17}+4\right)=10$$

$$y=\frac{1}{1}\;(17+3)=20$$

(ii) Let us assume arbitrarily $y=20$ On substituting this value of y in the above equation we get

$$20x-3x=170$$

whence $x=10$

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1 भाविनके यद्राशी विनष्टवर्गेन तत्प्रमाणानि ।
क वेदानि [illegible] भवति [illegible] ॥
[illegible] ।
[illegible] किं कृतं [illegible] ॥ —*BrSpSi XVIII 62-63*

2 [illegible] ।
[illegible] ॥ —*BrSpSi XVIII 61*

— o —

Reference

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CHAPTER X

Arabic and Indian Divisions of the Zodiac

It has long been a debated question whether the Indian and Arabian divisions of the zodiac had a common origin Sir William Jones thought that they had not, but Colebrooke holds a contrary view The coincidence, in the two-systems of division is so exact that he thinks, it could not be due to chance Colebrooke has discussed this point in details in one of his Papers entitled "On the Indian and Arabian divisions of the zodiac", *Asiatic Researches* Vol ix p 323–376, reproduced in the *Miscellaneous Essays*, Vol II, p 321 373, 1872

1 *Aśvinī* now the first *nakṣatra*, but anciently the last but one probably obtained its present situation at the head of the asterisms, when the beginning of the zodiac was referred to the first degree of Meṣa (the Ram) As measuring a portion of the zodiac it occupies the first 13°20′ of Meṣa and its beginning follows immediately after the principal star in the last *nakṣatra* Revati, reckoned by some exactly, by others nearly, opposite to the very conspicuous one, which forms the fourteenth asterism As a constellation, Aśvinī comprises three stars (Aries α, β, γ) figured as a horse-head and the principal, which is also the northern one is stated by all ancient authorities, in 10°N and 8°E from the beginning of the Meṣa

According to Arabs, the first manzil or lunar mansion is entitled *Sheratan* (by Persians *Sheratain*) and comprises two stars of the third magnitude on the head of Aries in lat 6°36′ and 7°51′ N and long 26°13′ and 27°7′. With the addition of a

third, also in the head of the Ram, the asterism is denominated *Ashrait*. The bright star of the second or third magnitude which is out of the figure of the Ram, according to Ulugh Beg, but on the nose according to Hipparchus, cited by this author from Ptolemy, is determined *Nātih* : It is placed in lat. 9°30'N and long. 1 0° 43', and is apparently the same with the principal star of the Indian asterism; for Muhammad of Tizin, in his table of declination and right ascension, expressly terms it the first star of the *Sheratain*.

2. *Bharaṇī*, the second asterism, comprises three stars (35, 39, 41 Aries) figured by the *yoni* or *pudendum muliebre* and *the principal and southern star of this naksatra is placed in* 12°N. On the Arabian system, the second manzil, entitled *Butain* is placed by Ulugh Beg in lat. 1°12' and 3°12', and this cannot possibly be reconciled with the Indian constellation. But Muhammad of Tizin assigns to the bright star of *Butain* a declination of 23°N exceeding by nearly 2° the declination allotted by him to *Nātih* or his first star in *Sheratain*. This agrees with the difference between the principal stars of *Aśvinī* and *Bharaṇī*; and it may be inferred, that some among the Mohammadan astronomers have concurred with the Hindus, in referring the second constellation to stars that form Musca.

3. *Kṛttikā*, *now the third*, *formerly the first*, *nakṣatra* consists of six stars figured as knife or razor, and the principal and southern star is placed in 4½ or 5°N and in 65 sixths of degrees (or 10°50') from its own commencement (cf. the *Sūryasiddhānta*), or 37° 28' to 38° from the beginning of the Meṣa (the *Siddhānta-Śiromaṇi* or the *Graha Lāghava*) respectively. This longitude of the circle of declination corresponds nearly with that of the bright star in the Pleiades, which is 40° of longitude distant from the principal star of *Revatī*.

The stars indicated by Ulugh Beg for *Thurayya*, also correspond exactly with the Pleiades.

4. *Rohiṇī*, is the fourth *nakṣatra*, the Arabic name for the fourth mansion is *Debarān* (or with the article *Aldebarān*). It corresponds to the bright star called the Bull's eye, and which is unquestionably *the same with the* principal and eastern star of *Rohiṇī* placed in 4½° or 5°S and 49½°E by the Hindu writers on Astronomy. This *nakṣatra* is

figured as a wheel cart and comprises five stars, out of the seven which the Greeks named the Hyades The Arabs however like the Hindus, reckon five stars only in the asterism Sir William Jones supposes them to be in the head and neck of the Bull they probably are α, ρ, γ, δ, ε Tauri agreeably to Mons Bally s conjecture

5 *Mṛgaśirā* the fifth *nakṣatra*, represented by an antelope head contains three stars, the same which constitute the fifth lunar mansion *Hakāh*, for the distance of 10°S assigned to the northern star of this *nakṣatra* will agree with no other but one of the three in the head of Orion The difference of longitude (24° to 25½°) from Kṛttikā corresponds with sufficient exactness and so does the longitude of its circle of declination (62° to 63°) from the end of Rewati since the true longitude of λ Orionis from the principal star in Revati (ζ piscium) is 63¼°

6 *Ārdrā* the sixth *nakṣatra* consists of a single bright star described as a gem and placed in 9°S (by some in 11°) and at the distance of 4½ to 4° in longitude from the last asterism This indicates the star in the shoulder of Orion (α orionis) The sixth lunar mansion is named by the Arabs as *Hanah* and comprises two stars in the feet of the second twin according to Ulugh Beg though others make it to be a shoulder Mohammad of Tizin allots five stars to this constellation, and the Kāmūs among various meanings of Hanāh says that it is a name for five stars in the left arm of Orion remarking also, that the lunar mansion is named *Tahāyi* comprising three stars called Tahyāt Obviously here the Indian and Arabian asterisms are irreconcileable

7 *Punarvasū* (used in a dual number) is the seventh *nakṣtra*, and is represented by a house or even a bow and it includes four stars among which the principal and eastern one is 30° or 32° from the fifth asterism but has been placed by all authorities in 6°N This agrees with (β Geminorum) one of the two stars in the heads of the Twins, which together constitute the seventh lunar mansion *ziraā* according to Mohammad of Tus and Mohammad of Tizin and other Arabian authorities The seventh lunar mansion of Arabs is named *ziraā ul asād* according to Jauhari and other cited by Hyde in his Commen

tary on Ulugh Beg, and that the *Kāmūs* makes this term to be the name of eight stars in the form of a bow.

8. *Puṣya*, the eighth asterism, is described as an arrow, and consists of three stars the chief of which being also the middle most, has no latitude, and is 12° to 13° distant from the seventh asterism, being placed by Hindu astronomers in 106° of longitude. This is evidently δ Cancri; and does not differ widely from the eighth lunar mansion *Nethrah*, which according to Ulugh Beg and others consists of two stars, including the nebula of Cancer. The Indian constellation comprises two other stars besides δ Canceri, which are perhaps γ and β of the same constellation.

9. *Aśleṣā*, the ninth asterism, contains five stars figured as a potter's wheel, and of which the principal or eastern one is placed in 7°S, and according to different tables, 107,° 108° or 109° E. This appears to be intended for the bright star in the southern claw of Cancer (α Canceri), and cannot be reconciled with the lunar mansion *Tarf* or *Tarfah*, which comprises two stars near the lion's (*siṁha*) eye, the northernmost being placed by Mohammad of Tizin in 24° of N. declination.

10. *Magha*, the tenth asterism, contains like the last, five stars; but which are figured as a house. The principal of the Southern one has no latitude; and according to all authorities, has 129° longitude. This is evidently Regulus (α Leonis) : which is exactly 129¼° distant from the last star in *Revati* The tenth lunar mansion of Arabians is *Jebhah*, which comprises three (some say, four) stars, nearly in the longitude of the lion's heart. In this instance, therefore, the Indian and Arabian divisions of the zodiac coincide. This *nakaṣtra* consists of α γ ζ η and ν Leonis.

11. *Pūrva-Phālgunī* is the eleventh *nakṣatra* and is represented by a couch or bedstead: it consists of two stars determined by the place of the chief star (the northernmost, according to the *Sūrya-Siddhānta*) in 12°N and 144°E, or according to Brahmagupta, the *Śiromaṇi* and the *Grahalāghava* 147° or 148° E. They are probably δ and θ Leonis. The Arabian name for this lunar mansion is *zubrah* or *Khertān*.

It may be mentioned here that Brahmagupta and Bhāskara selected the southern for the principal star, while the *Sūrya-Siddhānta* took the northern Hence the latitude stated by several Hindu authorities is the mean between both stars, and the difference of longitude, compared to the preceding and subsequent asterisms, may be exactly reconciled upon this supposition

12 *Uttara Phālguni* which is the twelfth *nakṣatra*, consists of two stars, and is figured as a bed or cot These stars are ascertained by the place of one of them (the northernmost) 13°N and 155° E. This indicates β Leonis the same which singly constitutes the Arabian Lunar mansion *Serfah*, though Mohammad of Tizin seems to hint that it consists of more than one star

13 *Hasta*, the thirteenth *nakṣatra* has the name and figure of a hand and is suitably made to contain five stars The principal one towards the west, next to the north-western star, is placed according to all authorities in 11° and 170° E This can only belong to the constellation Corvus, and accordingly five stars in that constellation (α, β, γ, δ and ε Corvi) The thirteenth lunar mansion of Arabs is *Awwa*, which is also described to contain five stars, situated under Virgo and so disposed as to resemble the letter *Alif* They are placed by Ulugh Beg in the wing Here obviously there is nothing common between the Hindu and Arabian specification of the asterism The agreement is only in the number of stars and in the longitude

14 *Citrā* the fourteenth *nakṣatra*, is figured as pearl It is placed by the *Sūrya-Siddhānta* in 2° S and 180° E. and by Brahmagupta the *Śiromani* and *Grahalaghava* in 1¾ or 2°S and 183°E. This agrees with the Virgin's spike (α Virginis) The same star constitutes the fourteenth lunar mansion of the Arabs named from it *Simac ul aazi*

15 *Svāti*, the fifteenth *nakṣatra* is represented by a coral bead The *Sūrya-Siddhānta*, Brahmagupta, the *Śiromani* and *Grahalaghava* all concur in placing it at 37°N They differ one degree in longitude of its circle of declination, three of them

making it 199° and the other 198° The Indian asterism totally disagrees with the lunar mansion *Ghafr* which is the fifteenth Arabian mansion and which consists of three stars in the Virgin s (*Kanyā*) foot, according to Ulugh Beg but in or near the balance (*Tulā*) according to others

16 *Viśākhā*, the sixteenth *nakṣatra* consists of four stars described as a festoon All the authorities place the principal and northernmost star in 1°, 1°20′ or 1°30 S and in 212° 212°5′ or 213° E The latitude seems to indicate the bright star in the Soutehrn Scale (α Librae) though the longitude disagrees (suggest ing possibly a remote star κ Librae) Colebrooke suggests the four stars to be α ν ι Librae and γ Scorpii The sixteenth lunar mansion according to Arabs is *Zubanah* or *Zubāniyah* according to Mohammad of Ti-In the bright star in the northern s ale (β Librae)

17 *Anurādhā* the seventeenth *nakṣatra* consists of four stars and is described as a row of oblations in a right line Its chief or middlemost star is placed in 3° or 2° or 1°45 S and in 224° or 224°5 E. thus placing it near the head of the Scorpion (*Vṛścika*) (δ Scorpionis) and the asterism comprises β δ π and ρ Scorpionis. The seventeenth lunar mansion of Arabs is called *Iklīl* or *Iklilu l-jebhah* which is said to contain 4 3, or 6 stars lying in a straight line Those assigned by Ulugh Beg for this mansion are β δ ν and π Scorpionis Thus here the Indian and Arabian astronomers both concur exactly

18 *Jyeṣṭha* the eighteenth *nakṣatra* comprises three stars figured as a ring The principal and middlemost star is placed in 4°.3½° or 3° S and in 229°.229°5 or 230°E this position indicates Antares or the Scorpion s heart (α Scorpionis) which is also the eighteenth lunar mansion named *Kalb* or *Kalbul ākrab* The three stars of Indian asterism may be α, σ and τ Scorpionis

19 *Mūla* the nineteenth *nakṣatra*. is represented by a lion s tail and it contains eleven stars. of which the characteristic one the easternmost. is placed in 9° 8½° or 8° S and in 241° or 242° E This probably (not exactly) indicates ν Scorpionis This agrees with the eighteenth lunar mansion of Arabs known as *Shaulah* consisting of two stars near the Scorpion's

sting The Hindu asterism probably includes all the stars in the Scorpion's tail (ε, μ, ζ, η, θ, ι, κ, λ, ν and υ Scorpionis)

20. *Pūrva-Āṣāḍha*, the twentieth *nakṣatra* is figured as an elephant's tooth or as a couch, and it consists of two stars, of which the most southern one is placed in 5⅓°, 5½ or 5° S and 254° or 255° E This corresponds well with δ *Sagittarii* and which also corresponds with the twentieth lunar mansion of Arabs called *Naaim* The Arabian mansion consists of four or according to some eight stars The Indian *nakṣatra* corresponds to δ and ε *Sagittarii*

22 *Uttara Āṣāḍha*, the twenty first *nakṣatra*, is represented by a couch or by an elephant s tooth The principal or the most northerly star is placed in 5° S and 260° or 261° E agreeing *with a star in the body of Sagitterius* (τ *Sagittarii*), *and the* other star is perhaps the one marked ζ The Arabian lunar mansion corresponding to it is *Baldah*, consisting of six stars. two, of which are placed by Mohammad of Tizin in declination 21° and 16° One of these must be a star in the head of Sagittarius Some authors on the contrary, describe the lunar mansion as destitute of stars Here the Arabs and Hindus do not show reconciliation

22 *Abhijit* the twenty-second asterism, consists of three stars figuring as a triangle or as a nut of floating Trapa (in modern Indian astronomy, it does not occupy an equal portion of the ecliptic with other *nakṣatras*) Its brightest star is very remote from the zodiac, being in 60° or 62° N The longitude of its circle of declination is 265°, 266° 40' or 268° according to different authorities The corresponding lunar mansion of Arabs is *Zabih*, consisting of two stars (according to some, four) in the horns of Capricorn This totally disagrees with Indian asterism

20 *Śravana* the twenty-third *nakṣatra* is represented by three footsteps and contains three stars of which the middlemost is placed in 30° N (all authorities agree) and longitude 280° (*Sūrya Siddhanta*) or 278° (Brahmagupta and *Śiromaṇi*), or 275° (*Grahalaghava*) The assigned latitude indicates the bright star in the Eagle whence the three may be inferred to be α, β

and γ Aquilae. According to Arabs, the twenty-third lunar mansion is *Balā*, which consists of two stars in the left hand of Aquarius. Here again Arabian and Hindu divisions are at variance.

24. *Dhaniṣṭhā*, the twenty-fourth *nakṣatra*, is represented by a drum or tabor. It comprises four stars, the westernmost of which is placed in 36° N and according to Brahmagupta, *Śiromaṇi* and the *Sūrya-Siddhānta* in 290° E (*Grahalāghava* gives 286°). This longitude of the circle of declination and the distance of the star on it from the ecliptic indicate the Dolphin : and the four stars are α, β, γ and δ Dolphini. The corresponding lunar mansion of Arabs is *Saud*, which comprises two stars in Aquarius (β and ζ Aquarii). Here again the two divisions disagree completely.

24. *Śatahbiṣak*, the twenty fifth *nakṣatra*, is a cluster of 100 stars figured by a circle, The principal or the brightest has no latitude; or only a third, or at utmost half, a degree of south latitude; and longitude 320°. This best corresponds with λ Aquarii. According to Arabs, the twenty-fifth lunar mansion is known as *Akhbiyah* which consists of three stars only, placed in the wrist of the right hand of Aquarius. However, it appears from Ulugh Beg's tables, as well as from Mohammad of Tīzīn's, that four stars are assigned to this mansion. The Indian and Arabian systems of division differ considerably but less widely according to some.

26. *Pūrva-Bhādrapada*, the twenty sixth *nakṣatra*, consists of two stars represented by a couch or bed, or else by a double headed figure, one of which is placed in 24° N and 325° or 326° E. The only conspicuous star nearly in that position is the bright star in Pegasus (α Pegasi) and the other may be the nearest considerable star in the same constellation (ζ Pegasi). The twenty-sixth Arabian lunar mansion is *Mukaddim*, consisting of two brightest stars in Pegasus (α and β). Here the Indian and Arabian divisions show concurrence.

27. *Uttar-Bhādrapada* the twenty-seventh *nakṣatra*, consists of two stars, figured as a twin or a person with double face, or else as a couch. The position of the most northerly of the two

is in 26° or 27°N and 337° E. which probably *indicates the bright* star in the head of Andromeda and the other star to be the one in the extremity of the wing of Pegasus (γ Pegasi) This exactly agrees with the twenty-seventh lunar mansion of Arabs named as *Muakkher* Ulugh Beg assigns those stars to it

28 *Revati*, the twenty-eighth *nakṣatra* comprises thirty two stars figured as a tabor The princ pal star is the southern most one it has no latitude and two of them assert no longitude. *but some make it ten minutes short of the origin of the ecliptic viz* 359° 50 *This clearly marks the star on the ecliptic in the* string of the Fishes (ζ Piscium) The ascertainment of this star is important in regard to the adjustment of the Hindu sphere. The Arabic name for this mansion is *Risha*. signifying a cord But the constellation as described by Jauhari and cited by Golius consists of a multitude of stars in the shape of a fish and termed *Betnu lhut* in the navel of which is th lunar mansion Moham mad of Tizin also makes this lunar mansion to be the same with *Betnu lhut* which appears. however to be the bright star in the *girdle of Andromeda (β Andromedae) though others describe it as the northern fish extending however to the horns of Ram* The lunar mansion and the Indian asterism therefore are not reconcileable in this last instance.

I leave it to the readers to draw an inference as to the concurrence of the divisions of zodiac in Indian and Arabian systems I would *personally agree with Sir William Jones that the agreements are by chance* Arabs derived the idea of dividing zodiac in 27 or 28 mansions from Ind ans or may have got it from Greeks, and then they proceeded in their own way for details I do not agree with those scholars who sometimes state that the Hindus took the hint of dividing the ecliptic from Greeks The *Atharvaveda* devotes a number of Sūktas or hymns on Nakṣatras, and I have shown elsewhere that inspired by these hymns Gārgya was the first Ṛṣi *who detailed out the nakṣatras This happened much before Greeks developed even their first notions of astronomy* While the concept of 27 *nakṣatras* is Vedic and most ancient and of purely Indian origin the concept of 12 Rāśis (signs) or twelve constellations is proba bly inspired from Greeks [The names Kanya, (virgo) *Tula*

(Libra),Vṛścika (Scorpio), Dhanu, (Sagittarius), Makara, (Capricorn), Kumbha (Aquarius), Mīna (pisces), Mesa (Aries), Vrsa (Taurus), Mithuna (Gemini), Karka (Cancer), and Simha (Leo) were not used for Rāśis or signs in the Vedic times]. I shall conclude this description with a passage from Colebrooke :

> The result of comparison shows, I hope satisfactorily, that the Indian asterisms, which mark the divisions of the ecliptic, generally consist of nearly the same stars, which constitute the lunar mansions of the Arabians : but in a few instances, they essentially differ. The Hindus have likewise adopted the division of the ecliptic and zodiac into twelve signs or constellations, agreeing in figure and designation with those of the Greeks; and differing merely in the place of the constellations, which are carried on the Indian sphere a few degrees further west than on the Grecian. That the Hindus took the hint of this mode of dividing the ecliptic from the Greeks, is not perhaps altogether improbable; but if such be the origin of it they have not implicitly received the arrangement suggested to them, but have reconciled and adapted it to their own ancient distribution of the ecliptic into twenty-seven parts.
>
> In like manner, they may have either received or given the hint of an armillary sphere as an instrument for astronomical observation ; but certainly they have not copied the instrument which was described by Ptolemy, for the construction differs considerably.

Names, Shapes, and Number of the Stars of the Nakṣatras

The *Muhūrta-cintāmaṇi* provides a list of shapes associated to the nakṣatras (*MuC.* II 59-60). In this list we are giving the number of stars as indicated by Varāhamihira, Brahmagupta and Lalla. The identification given here is as indicated by E. Burgess, in his Translation of the *Sūrya Siddhānta* 1935 p. 378, (Calcutta). This table has been reproduced here from the *Mahābhāskarīya* of Bhāskara I, edited by K.S Shukla.

CHAPTER XI

Brahmagupta's Astronomy : Its Highlights

Beginning or Starting Point

Very often in Indian astronomy we come across a term *ahargaṇa* (literally meaning collection of days) which means the number of mean civil days elapsed at mean Sunrise at Laṅkā on a given lunar day (*tithi*) since the beginning of *Kaliyuga*. It is the beginning of *Kaliyuga*, which is taken as the starting point for the reckoning of *ahargana*. This happened on Friday, February 18, B.C. 3102 at mean sunrise at Laṅkā when the Sun, Moon and the planets are supposed to have been in conjunction at the first point of the *nakṣatra* *Aśvini* (which is a fixed point situated near the star ζ Piscium). According to Āryabhaṭa and Bhāskara I, the duration of Kaliyuga is 1,080,000 solar years. Four times this (4,320,000) is the duration in solar years of a bigger unit called *Mahāyuga* or even *yuga*.

Laṅkā in Indian astronomy is a hypothetical place where the meridian of Ujjain (latitude 23° 11' N, longitude 75° 52' E from Greenwich) intersects the equator. It is one of the four hypothetical cities on the equator called Laṅkā, Romaka, Siddhapur and Yamakoṭi (or Yavakoṭi). The *Sūrya siddhānta* describes Laṅkā as a great city (mahāpuri) situated on an island to the south of Bhāratavarsa[1]. The present Ceylon is not the

1 समन्तान्मेरुमध्यात्तु तुल्यभागेषु तोयधेः ।
द्वीपेषु दिक्षु पूर्वादिनगर्यो देवनिर्मिताः ॥
भूवृत्तपादे पूर्वस्यां यवकोटीति विश्रुता
भद्राश्ववर्षे नगरी स्वर्णप्राकारतोरणा ।
याम्यायां भारतेवर्षे लङ्का तद्वन्महापुरी ॥

(Cont. on page 290)

astronomical Lankā, as it is about six degrees to the north of equator. The astronomical Lankā is mentioned by Brahmagupta in the beginning of his very first Chapter[1].

According to Brahmagupta all the four yugas of a *Caturyuga* or *mahāyuga* are not of the equal duration :

> *Kaliyuga* is of 432,000 years. *Dvāpara* of 864,000 years. *Tretā* of 1,296,000 years and *Kṛtayuga* of 1,728,000 years; total of the four is 4,320,000 years. Āryabhaṭa regards all *yugas* of equal duration, 1,080,000 years[2].

The Saka era, which is usually used in Indian astronomy for the reckoning of years commenced 3179 years after the beginning of *Kaliyuga*.

The number of lunar months in a *yuga* does not coincide with the number of solar months. Thus we have the conception of the *Intercalary months* : the number of intercalary months in a *yuga* denotes the excess of the number of lunar months in a *yuga* over the number of solar months in a *yuga*. Thus in a *yuga* we have

Lunar months	53,433,336
Solar months	51,840,000
Intercalary months	1,593 336
Lunar days	1,603,000,080
Civil days	1,577,917,500
Omitted lunar days	25,082,580

The number of omitted lunar days in a *yuga* is equal to the number of lunar days in a *yuga* minus the number of civil days in a *yuga*.

(Cont. from page 289)

पश्चिमे केतुमालाख्ये रोमकाख्या प्रकीर्तिता ।
उदक्सिद्धपुरी नाम कुरुवर्षे प्रतिष्ठिता ॥ —*Sūrya*. XII. 36-39

1. चैत्रसितादेरुदयाद्भानोर्दिनमासवर्षयुगकल्पाः ।
सृष्ट्यादौ लंकायां सम प्रवृत्ता दिनेऽर्कस्य ॥ —*BrSpSi*. I. 4

2. युगदशभागो गुणितः कृतं चतुर्भिस्त्रिभिर्युग्मैस्त्रेता ।
द्विगुणो द्वापरमेकेन संगुणः कलियुगं भवति ॥
युगपादानार्यभटश्चत्वारि समानि कृतयुगादीनि ।
यदभिहितवान् न तेषां स्मृत्युक्तसमानमेकमपि ॥ —*BrSpSi*. I. 8-9

Units of time

For the measurements of durations it is necessary to have units of time Brahmagupta gives the following units [1]

6 prānas or Asus	=1 Rksa vinādikā or naksatra vighaṭikā or one *pala* (24 seconds)
60 palas	=1 ghatikā (24 minutes)
60 ghaṭikās	=1 divasa or dina (day) (24 hours)
30 dinas	=1 māsa (month)
12 māsas	=1 varsa or year

Similar to the divisions of time we have the divisions of an arc [2]

Vikalā (or viliptā or viliptikā)	=second of arc
60 vikalās	=1 kalā (minute of arc)
60 kalās	=1 amśa (degree of arc)
30 amśas	=1 rāśi
12 rāśis	=1 bhagana (complete circle 360°)

Unlike Āryabhaṭa and others who take kali dvāpara tretā and krta of equal number of years Brahmagupta regards kali consisting of 432 000 years dvāpara twice of it consisting of 864 000 years tretā thrice of Kali consisting of 1 296 000 years and krta four times of kali conisting of thus 1 728 000 years all the four to be making a *yuga* of 4 320 000 years [3] Further in the beginning of kṛta there is a *sandhya* of 1728000/12 years (=144 000 years) and at the end of Krta, there is a sandhyāmśa of 144 000 years similarly in the beginning of tretā we have a

1 प्राणैर्विनाडिकार्क्षी षड्भिर्घटिका षष्ट्या ।
घटिका षष्ट्या दिवसो दिवसानां त्रिंशता मासः ॥ —*BrSpSi* I 6

2 मासा द्वादश वर्षं विकलालिप्तांशराशिभगणात् ।
क्षेत्र विभागस्तुल्य कालेन विनाडिकाद्येन ॥ —*BrSpSi* I 6

वर्षं द्वादश मासास्त्रिंशद्दिवसो भवेत्स मासस्तु ।
षष्टि नाड्यो दिवसः षष्टिस्तु विनाडिका नाडी ॥ —*Ārya* III 1

गुर्वक्षराणि षष्टिर्विनाडिकार्क्षी षडेव वा प्राणाः ।
एवं काल विभागः क्षेत्र विभागस्तथा भगणात् ॥ —*Ārya* III 2

3 खचतुष्टयरदवेदा रवि वर्षाणां चतुर्युगं भवति ।
सन्ध्या सन्ध्यांशैः सह चत्वारः पृथक् कृतादीनि ॥
युगदशभागो गुणितः कृतं चतुर्भिस्त्रिभिर्द्वाभ्यां त्रेता ।
द्विगुणो द्वापरमेकेन सगुणं कलियुगं भवति ॥ —*BrSpSi* I 7 8

sandhyā of 1,296,000/12; i. e. 108,000 years and at the close of tretā a sandhyāṁśa of 108,000 years. Again, in the beginning of dvāpara we have a sandhyā of 864,000/12, i. e. 72,000 years and at the close of dvāpara a sandhyāṁśa of 72000 years; and similarly at the beginning a sandhyā and at the close a sandhyāṁśa of 432,000/ 1, i. e. of 36,000 years in the case of kali. In this respect Brahmagupta appears to follow Manu, the first author or giver of law. He regards further the following divisions of time :[1]

71 yugas =1 manu
14 manus =1 kalpa

Again, in the beginning, at the middle and at the close of each manu, there are sandhis, each equal to the measure of krta. Thus, taken as a whole

$$
\begin{aligned}
1 \text{ kalpa} &= 71\times 14 \text{ yugas}+15 \text{ sandhyā-sandhyāṁśa} \\
&= 994 \text{ yugas}+15\times \text{duration of krta} \\
&= 994 \text{ yugas}+15\times(4\times 432{,}000) \text{ years} \\
&= 994 \text{ yugas} + 6 \text{ yugas} = 1000 \text{ yugas} \\
&= 1 \text{ Brahma-dina (Brahmā's day)}
\end{aligned}
$$

Thus Brahmā's day is regarded as 1 kalpa or one thousand caturyugis or 1000 *yugas* or the same as 1000 *mahāyugas*).

Āryabhaṭa regards a manu to consist of 72 yugas and therefore a kalpa according to him would be of 14×72 yugas, or 1008 yugas.[2] Since in the foreign Siddhāntas like Romaka, there is no reference to yuga, manu and kalpa, Brahmagupta regards these systems to be unauthoritative.[3]

We have said that our starting point was the beginning of Kaliyuga, Friday February 18, B.C. 3102, at mean rise at Laṅkā, when the Sun, Moon and the planets are supposed to have been *in conjunction at the first point of the Nakṣatra Aśvinī.* This type of conjunction would again happen after a period of *kalpa.*

1. मनुरेकसप्ततियुगः कल्पो मनवश्चतुर्दश मनूनाम् ।
आद्यन्तान्त सन्धिषु कृतमानोऽस्माद् युग सहस्रम् ॥ —*BrSpSi.* I. 10

2. दिव्यं वर्षं सहस्रम् ग्रहसामान्यं युगं द्विषट्क गुणम् ।
अष्टोत्तरसहस्रं ब्राह्मो दिवसो ग्रहयुगानाम् ॥ —*Ārya* III. 8

3. युगमन्वन्तरकल्पाः कालपरिच्छेदकाः स्मृतावुक्ताः ।
यस्मान्न रोमके ते स्मृतिबाह्यो रोमकस्तस्मात् ॥ —*BrSpSi.* I- 13

Planet or a body	Bhagaṇas
Ravi or Sun	4 320 000 000
Budha or Mercury	4 320 000 000
Śukra or Venus	4 320 000 000
Candra or Moon	57 753 300 000
Kuja or Bhauma or Mars	2 296 828 522
Budha śīghrocca	17 936,998 984
Bṛhaspati or Jupiter	364 226 455
Śukra śīghrocca	7 022,389 492
Śani or Saturn	146 567 298
Arka or Ravi mandocca	480
Candra mandocca	488 105,858
Kuja or Bhauma mandocca	292
Budha mandocca	332
Bṛhaspati or Jīva mandocca	855
Śukra mandocca	653
Śani mandocca	41
Candra pāta	232 311 168
Kuja or Bhauma pāta	267
Budha pāta	521
Bṛhaspati or Guru pāta	63
Śukra pāta	893
Śani pāta	584

By pāta is meant the ascending node of a planet's orbit (on the ecliptic)

In a kalpa the number of *bha bhramas* (sidereal days) or also known as *bha parivartas* is 51 040 000 000 If we subtract out from this number the *bhagaṇa* of the Sun we get what is known as *ku-dinas* or *Sāvana* days or the solar or sacrificial days (51 040 000 000—4 320.000 000 46 720.000.000 Sāvana days or kudinas)

In a kalpa the number of Ravi bhagaṇas also correspond to the number of solar years (Saura varṣas) i e 4 320 000 000 this number multiplied by 12 gives the number (i.e 51,840 000,000) of solar months.

The difference between the candra bhagaṇas and the Ravi bhagaṇas in a kalpa gives the number of lunar months (Cāndra

māsa) in a kalpa (57 753 300 000-4 320 000 000=53.433 300 000 lunar months)

By subtracting the number of solar months from the number of lunar months in a kalpa one gets the number of *adhi masas* (additional months) 53 433 300 000 51 840 000 000= 1 593 300 000 adhimāsas. This multiplied by 30 gives the number of lunar days (śaśi-divasa) in a kalpa 53 433 300 000×30= 1 602 999 000 000 lunar days The difference between the lunar days and *kudinas* in a kalpa gives the number of *avama dinas* in a kalpa 1 602 999 000 000 45 720 000 000—1 556 279 000.000[1]

Brahmagupta calculates out the sṛṣṭi samvatsara or the Creation Era during his year of composition of the Treatise He says Six *manus have gone in the kalpa the seventh manu is* now running of which have lapsed 27 caturyugis of the twenty eighth caturyugi the three yugas kṛta dvāpara and tretā have gone by and also of the present kaliyuga 3179 years have lapsed The total period thus lapsed on calculation comes to be 1 972 947 179 years [2]

> Total Period=6 manus+7 manu—sandhis+27 yugas +kṛta+dvāpara+tretā+3179 years of kali
> =(6×71×4,320 000 years) +(7×4×432 000 years)+ (27×4 320 000 years)+(1 728 000+1 296 000+864 000)+ 3.179=1 972 947 179 years
> =1 840 320 000+12 096 000+116 640 000+3 888 000+ 3 179=1 972 947 179 years

Calculation of Ahargana The method of calculating ahargana (number of days elapsed since the beginning of kaliyuga)

---

1 [illegible] ।
[illegible] ॥
[illegible] ।
[illegible] ॥
[illegible] ।
[illegible] ॥ —*BrSpSi* I 22 24

2 [illegible] ।
[illegible] ॥
[illegible] ।
[illegible] ॥ *BrSpSi* I 26-27

has been given by Brahmagupta and Bhāskara I is almost identical. The rule given in the *Brāhmasphuṭasiddhānta*[1] may be compared with the following given by Bhāskara I in the *Laghu-Bhāskarīya*:

> Add 3179 to the (number elapsed) years of the Śaka era, (then) multiply (the resulting sum) by 12, and (then) add the (number of lunar) months (expired) since the commencement of Caitra. Set down (the result thus obtained) at (two) separate places; multiply (one) by (the number of) intercalary months in a *yuga*, which are 1,593,336 in a yuga : and divide (the product) by 5,184 × 10,000 (i.e.) by 51,840,000). Add the (resulting complete) intercalary months to the result *placed at the other place. Then multiply (that sum)* by 30 and (to the product) add the (lunar) days (i.e. *tithis*) expired of the current month. Set down (the result thus obtained) in two places; multiply (one) by the (number of) omitted lunar days in a *yuga* i.e. by 25,082,580 and divide by 1,603,000,080. The resulting (complete) omitted lunar days when subtracted from the result put at the other place give the (required) *ahargaṇa*. The remainder obtained on dividing (the *ahargaṇa*) by 7 gives the day beginning with Friday at sunrise (at Laṅkā)[2]

---

1. कल्पगताब्द द्वादशघातश्चैत्रादिमास युक्तोऽधः ।
गुणितो युगाधिमासै रविमासाप्ताधिमास युतः ॥
त्रिंशद् गुणास्तिथियुत्य पृथग् युगावमगुणो युगेन्दु दिनैः ।
भक्तः फलावमोनोऽर्क सावनाहर्गणोऽर्कादिः ॥ — *BrSpSi*. I. 29-30

2. नवाद्र्येकाग्नि संयुक्ताः शकाब्दा द्वादशाहताः ।
चैत्रादिमास संयुक्ताः पृथग् गुण्या युगाधिकैः ॥
ते च षट् त्रिकरामाहित नव भूतेन्दवो युगे ।
भागहारोऽब्धि वस्वेक शरास्तु युताहताः ॥
अधिमासाः पृथक्स्थेषु प्रक्षिप्य त्रिंशताहते ।
युक्तवा दिनानि यातानि प्रतिराश्य युगावमैः ॥
संगुण्या वसुषट्पञ्चद्व्यष्टशून्यशरात्रिभिः ।
छेदः खषड्वियद् व्योमत्र्यग्नि रसेन्दवः ॥
लब्धान्यवमशेषाणि तेषु शुद्धेष्वहर्गणः ।
वारः सप्तहृते शेषे शुक्रादिर्भास्करोदयात् । —*LBh I*. 4-8

Addendum The mean lunar day (*madhyama tithi*) may, however, differ from a true lunar day (*spaṣṭa tithi*) by one, so that the *ahargaṇa* obtained by the above process may sometimes be in excess or defect by *one* To test whether the *ahargaṇa* (obtained by the above process) is correct, it is divided by seven and the remainder counted with Friday. If this leads to the day of calculation, the *ahargaṇa* is correct, if it leads to the preceding day, the *ahargaṇa* is in defect, and if that leads to the succeeding day, the *ahargaṇa* is in excess When the *ahargaṇa* is found to be in defect it is increased by *one*, when it is found to be in excess, it is diminished by *one* (K S Shukla, *MBh* p 4–5)

Example—Calculate the *ahargaṇa* on October 1,1955

From Indian Calendar we find that October 1,1965 falls on Friday 7th lunar day (tithi) in the light half of the 7th month Āśvina in the Saka year 1887 (elapsed) Let us proceed as follows

Adding 3,179 to 1,887 we get 5 066 (1)

Multiplying this by 12 and adding 6 (i e. the number of lunar months elapsed since the beginning of Caitra) we get 60,798 .. ... (2)

Multiplying this by 1,593,336 and dividing the product by 51,840 000 we get 1,868 as quotient (The remainder is discarded as unnecessary) (3)

Adding this number (i e. 1,868) to the previous one (i e) 60,798) we get 62,666 (4)

Multiplying this by 30 and adding 6 (i e the number of lunar days elapsed since the beginning of the current month) to the product we get 1,879 986 (5)

Multiplying this by 25,082,580 and dividing the product by 1,603 000,080 we get 29 416 as the quotient (The remainder is discarded as not necessary) (6)

Subtracting this number (i e 29,416) from the previous one (i e 1,879 986) we get 1,850,570 (7)

This is the required *ahargaṇa* Since division by 7 leaves

1 as the remainder we subtract one from it, and get 1 850 569 as the correct *ahargaṇa* for the day

An Alternative Rule for Ahargaṇa

Both Bhāskara I and Brahmagupta give an alternative rule for calculating out *ahargaṇa*[1]

> Multiply the number of (solar months) elapsed since the beginning of kaliyuga by the number of lunar months (in a *yuga*) and divide by the number of solar months (in a *yuga*) Reduce the quotient to days (and add the number of lunar days elapsed since the beginning of the current lunar month), then multiply by the number of civil days (in a *yuga*) and divide by the number of lunar days (in a *yuga*) the quotient denotes the *ahargana*

Mean Longitude of a Planet

(1) The mean longitude of a planet in *revolutions* is given by the expression (Brahmagupta[2] and also Bhāskara[3])

---

1 शशाङ्कमासैरभिताडितान् हरेद्गतान् मासान् यथार्कसम्भवैः ।
दिनीकृतान् भूमिदिनैर्हतान् दिनैर्विभज्य लब्धः शशिजैरहर्गणः ॥ MBh I 7
युगगतशशिमासवधाद्रविमासाप्तं दिनीकृतं सदिनम् ।
भूदिनगुणितं शशिदिनहृल्लब्धमहर्गणः सैकः ॥ —BrSpSi XIII 18

2 इष्टग्रहभगणगुणादहर्गणात् कल्पसावनैः हृतात् ।
भगणादि फलं मध्यो लङ्कायां भास्वदौदयिकः ॥ BrSpSi I 31

3 हन्यादिष्टान् यान् भगणान् द्युमादिनैर्लभामहे यान् कलियातवासरैः ।
इति प्रलब्धा भगणास्ततः क्रमाद् गृहांशलिप्ता विकलाः सतत्पराः ॥ —MBh I 8
पर्ययाहर्गणाभ्यासो ह्रियते भूदिनैस्ततः ।
लभ्यन्ते पर्यया शेषाद्राशिभागकलादयः ॥
भास्करैस्त्रिंशता षष्ट्या सह गुणय्य पृथक् पृथक् ।
तेनैव भागहारेण लभ्यन्तेऽर्कोदयावधेः ॥ —LBh I 15-17

(Divide the product of the revolution number of a planet and the *ahargana* by the (number of) civil days (in a *yuga*) thus are obtained the (number of) revolutions (performed by that planet) From the (successive remainders multipl ed respectively by 12,30 and 60 and divided by the same d visor (i e the number of civil days in *yuga*) are obtained the signs degrees and minutes etc (of the mean longitude of that planet) for (mean) sunrise (at Lankā)

$$\text{Mean longitude} = \frac{\text{revolution number of planet} \times \textit{ahargaṇa}}{\text{civil days in a } \textit{yuga}}$$

Similar expression is given by more recent Indian astronomers also

(ii) Mean longitude of desired planets in minutes

$$= \frac{\text{(mean longitude of the known planet in revolutions etc reduced to minutes)} \times \text{(revolution number of the desired planet)}}{\text{revolution number of the known planet}}$$

This rule is common to Brahmagupta[1] and Bhāskara I[2]

(iii) An alternative rule for deriving the mean longitude of the Moon from that of the Sun and vice versa has been given by Bhāsakara I and Brahmagupta both

> Multiply the *ahargaṇa* by the number of intercalary months in a yuga and divide (the product) by the number of civil days (in a yuga) the result is in the terms of revolutions etc. Add that to thirteen times the mean longitude of the Sun (This is the process) to obtain the mean longitude of the Moon[3]

Mean longitude of the Moon

$$= \frac{\text{(intercalary months in a } \textit{yuga}) \times \textit{ahargaṇa}}{\text{civil days in a } \textit{yuga}} \text{ revolutions}$$

+13 (Sun's mean longitude)

This expression may be rearranged to get the mean longitude of the Sun from the mean longitude of the Moon[1].

Mean longitude of the Sun

$$=\frac{1}{13}\Big[\text{mean longitude of the Moon} - \frac{(\text{intercalary months in a yuga})\times \textit{ahargaṇa}}{\text{civil days in a yuga}}\text{revolutions}\Big]$$

Calculating the Mean Longitudes of the Sun and the Moon without using Ahargana

Bhāskara I follows the method of Āryabhaṭa I and the same method more or less has been adopted by Brahmagupta in calculating the mean longitudes of the Moon and the Sun without the use of *ahargaṇa*. The method may be described thus.

> Reduce the years elapsed since the beginning of kaliyuga to months and add to them elapsed months of the current year. Then multiply the sum by 30 and add the product to the number of lunar days elapsed since the beginning of the current month. Multiply that sum by the number of intercalary months in a *yuga* and divide by the number of solar months in a *yuga* reduced to days; the quotient denotes the number of intercalary months elapsed. The remainder is the *adhimāsaśeṣa*. Multiply the complete intercalary months thus obtained by 30 and to the product add the number of solar days elapsed since the beginning of kaliyuga[2] : then multiply that sum by the number of omitted lunar days in a *yuga* and divide by the number of lunar days in a *yuga* ; the remainder obtained is the *avamaśeṣa* called *āhnika*. Then multiply the *avamaśeṣa*

1 कुमुद्वतीशा सुहृदोऽथवाऽऽगतं विशोध्य शेषस्य लवस्त्रयोदशः ।
स मध्यमार्को गणकैर्निरूप्यते गुरुप्रसादात्प्रति बुद्ध बुद्धिभिः ॥ *MBh*. I. 11-12

2. By the number of solar days here is meant the number obtained above by reducing the years elapsed since the beginning of kaliyuga to months, then adding to them the number of months elapsed since the beginning of the current year, then multiplying the sum by 30, and then adding to the product thus obtained the number of lunar days elapsed of the current month.

K S Shukla has provided the following rationale to the rule cited above.

The fraction of the intercalary month (obtained in the rule) $= \frac{\text{adhimāsaśesa}}{\text{solar days in a } yuga}$, in mean lunar months

$$= \frac{\text{adhimāsaśesa}}{\text{lunar days in a } yuga}, \text{ in mean solar months} \qquad (i)$$

The fraction of the omitted lunar day (obtained in the rule)

$$= \frac{\text{avamaśeṣa or āhnika}}{\text{lunar days in a } yuga}, \text{ in mean civil days}$$

$$= \frac{\text{avamaśeṣa}}{\text{civil days in a } yuga} \text{ in mean lunar days}$$

$$= \frac{\text{avamaśeṣa} \times 60}{\text{civil days in a } yuga}, \text{ in mean lunar } ghaṭis \qquad (ii)$$

The fraction of the intercalary month corresponding to the above fraction of the omitted lunar day

$$= \frac{(\text{intercalary months in a yuga}) \times (\text{avamaśesa})}{(\text{lunar days in a } yuga) \times (\text{civil days in a } yuga)}$$

in mean solar months (iii)

Adding (i) and (iii) and multiplying by 30 the total fraction of the intercalary month

$$= \left\{ \frac{\text{adhimāsaśesa}}{\text{lunar months in a } yuga} + \frac{(\text{intercalary months in a } yuga) \times (\text{avamaśesa})}{(\text{lunar months in a } yuga) \times (\text{civil days in a } yuga)} \right\}$$

in mean solar days. (iv)

Suppose that m lunar months and d lunar days have elapsed since the beginning of Caitra Then treating them as mean lunar months and mean lunar days m months and d days denote the time elapsed since the beginning of mean Caitra up to the beginning of the current lunar day (treated as mean lunar day) As (ii) is the interval, in mean lunar *ghaṭis*, between the beginning of the current lunar day and the mean sunrise on that day therefore

$$m \text{ months} + d \text{ days} + (ii)$$

denotes the time in mean lunar months, days ghaṭis[1] elapsed

1 1 hour = 2½ ghaṭis 1 ghaṭi = 60 vighaṭis 1 vighaṭi = 60 pravighaṭis

since the beginning of mean Caitra up to the mean sunrise on the current lunar day

Like wise

m months $+ d$ days $+$ (ii) $-$ (iv)

denotes the time in mean solar months days ghaṭis etc elapsed since the beginning of the current mean solar year up to the mean sunrise on the current lunar day[1]

Let M D G V and P denote respectively the mean solar months mean solar days mean solar ghaṭis mean solar vighaṭis and mean solar pravighaṭis elapsed since the beginning of the current mean solar year up to the mean sunrise on the current lunar day Then evidently mean longitude of the Sun

= M signs, D degrees, G minutes V seconds and P thirds

= (m signs and d degrees) + [minutes, seconds etc corresponding to (ii)] − [degrees minutes etc. corresponding to (iv)]

and mean longitude of the Moon

= 13 [m signs and d degrees + (minutes seconds etc corresponding to (ii)] − [degrees, minutes etc. corresponding to (iv)]

because

[(1/12) mean longitude of the Moon − mean longitude of the Sun

= m signs + d degrees + (minutes, seconds etc corresponding to (ii)]

(This equality is based on the fact that the left hand side denotes the mean lunar date also known as *madhyama tithi*)

A similar rule of these calculations of the mean longitude

1 Because (v) is equal to fraction of a lunar month between the beginning of Caitra and the beginning of the current mean solar year fraction of an intercalary month corresponding to the tithis elapsed up to the beginning of the current mean lunar day since the beginning of Caitra fraction of an intercalary month corresponding to the avamaśeṣa i.e the lunar portion between the beginning of the current lunar date and the following sunrise

of the Sun and the Moon without basing on *ahargaṇa* has also been given by Brahmagupta in the *Khaṇḍakhādyaka*[1].

Concordance of Working Rules

There has been a good deal of agreement on various rules of astronomical constants from the time of Āryabhaṭa I (499 A. D.) to the Bhāskara II (1150 A. D.) or even later to the days of Muniśvara (1620 A. D). Earliest concepts were formulated during the days of the Vedāṅga-Jyautisa and the Siddhāntas of Indian and the western origin, for example of Brahma, Vasiṣṭha, Pitāmaha, Romaka and Puliśa. We in this section are giving some important concordances which we find common in the writings of Brahmagupta and his predecessors, contemporaries and successors as listed below. The list is not exhaustive. Only a a few illustrations have been cited.

1. Ārya.—*Āryabhaṭīya* Āryabhaṭa I. 499 A. D.
2. BrSpSi. —*Brāhmasphuṭasiddhānta.* Brahmagupta, 628 A. D.
3. K. K.—*Khaṇḍakhādyaka.* Brahmagupta. 628 A. D.
4. KKu —*Karṇa-kutūhala,* Bhāskara II. 1150 A. D
5. LBh —*Laghu-Bhāskarīya.* Bhāskara I, 522 A. D

MBh.—*Mahā-Bhāskarīya.* Bhāskara I. 522 A. D.
MSi.—*Mahā siddhānta,* Āryabhata II, 950 A. D
PSi —*Pañcasiddhāntika,* Varāhamihira, 505 A. D.
ŚiDVṛ.—*Śiṣyadhīvṛddhida.* Lalla, 598 A. D.
SiSā.—*Siddhāntasārvabhauma.* Muniśvara, 1620 A. D.
SiŚe—*Siddhāntaśekhara.* Sripati, 1039 A. D.
SiŚi.—*Siddhānta Śiromaṇi.* Bhāskara II, 1150 A. D.
SūSi.—*Sūryasiddhānta,* Modern, 6th or 7th Century.

1 Rule for finding the mean longitudes of the Sun, Mercury and Venus . *BrSpSi* I. 44

Also *MBh.* 1. 31. *MSi* I 26 ; SiŚe II 42, 43 , *SiŚi.* I 1, (d) 15 , *SiSa* I 105 , *KKu* I 7

2. Rule for finding the mean longitude of the Moon's ascending node *BrSpSi*, XXV. 35

1. त्रिनवरसभक्तमवमावशेषमाप्तं दिनादि तत्सहितात् ।
अधिमासशेषकाच्च त्रिंशद् गुणितात्तु खदिग्भिः ॥
मासदिन प्रथमैक्य पृथक् त्रयोदशगुण दिन योनौ ।
द्वावप्येवं मध्यौ दास्याद्यर्कचन्द्रौ वा ॥ —KK. I. 11–12

Also *MBh*. I. 33 ; *ŚiDVr*. I. i. 52 (ii)

3. Rule for finding the mean longitude of the *Śighrocca* of Venus and also giving the additives for the *Śighrocca* of Mercury and Moon : *BrSpSi*. XXV. 36.

Also *MBh*. I. 35 ; *ŚiDVr* I. i. 57 (ii)

4. Rule for finding the mean longitude of the *Śighrocca* of Mercury : *BrSpSi* XXV. 34.

Also *MBh*. I. 36 ; *ŚiDVr*. I. i. 50 (ii)

5 Rule for finding the mean longitude of Saturn : *BrSpSi*. XXV. 35.

Also *ŚiDVr*. I. i. 52 (i) ; *MBh* I. 37.

6. Rule for finding the mean longitude of Mars : *BrSpSi*. XXV. 33.

Also *ŚiDVr* I. i 50 (i) *MBh*. I. 38

7. Rule for finding the mean longitude of Jupiter : *BrSpSi*. XXX. 35.

Also *MBh* I 39 ; *ŚiDVr*. I. i. 51 (i).

8. Rule for finding the distance of a place from the prime meridian *BrSpSi* I 36.

Also *MBh*. II. 3-4 , *LBh*. I. 25-26 ; *ŚiDVr* I 57-58 (i) ; *SiSa* I. 143-144

9. Rule for finding the directions . *BrSpSi*. III. 1.

Also *MBh* III. 2 , *SaSi*. III 1-4 ; *LBh*. III. 1: *SiDVr*. I. iii. 1 : *MSi* IV 1 2 ; *SiSe*. IV 1-3 ; *SiŚi*. I. iii 8-9.

Alternative rule *BrSpSi* III. 2.

Also *MBh* III. 3 ; *PSi*. XIV. 14 16 : *ŚiDVr*. I. iii. 2 ; *SiSe* IV 4

10. Rule for finding the latitude and colatitude and the zenith distance and altitude of the Sun *BrSpSi*. III. 10.

Also *MBh* III 5. *SaSi* III. 13 14. *LBh*. III 2-3 *ŚiDVr* I. iii 4-5 · *SiSe* IV. 7. *SiŚi* I iii 18

11 Rule for determining the declination, day—radius, earth sine and ascensional difference (for the Sun or a point on the ecliptic) *BrSpSi* II 55

Also *SaSi* II. 23 . *LBh* II 16. . I. ii *ŚiDVr*. 17. *SiSe*. III.

63 64 ; *SiŚi*. I. ii. 47 (ii) (For RSine of the Declination).

BrSpSi. II. 56 ; also *Ārya*. IV 24 ; *MBh*. III. 6; *LBh*. II. 17 ; *ŚiDVr*. I. ii. 18 ; *SiŚe*. III. 66 ; . *SiŚi* I. ii. 48 (For day-radius).

BrSpSi. II. 57-58 ; also *MBh*. III. 7 ; *LBh*. II. 18 ;

SūSi .II. 61; *ŚiDVr*. I. ii. 18 ; *SiŚe* III. 67 ; *SiŚi*. I. ii. 49 (i) (For the ascensional difference).

12. For finding the times of rising of the sāyana signs at the equator : *BrSpSi*. III. 15.

Also *MBh*. III. 9 ; *SūSi*. III. 42–43 : *ŚiDVr*. I. iii. 8 ; *SiŚe*. IV. 15 ; *SiŚi* I. 11. 51.

13. Rule for finding the ascensional differences of the Sāyana signs Aries, Taurus and Gemini : *KK*. 1. 21.

Also *MBh*. III. 8 ; *PSi*. III. 10 ; *ŚiDVr*. 1. XIII. 9 ; *SiŚi*. I. ii. 50-51.

14. Rule for the determination of the meridian zenith distance and meridian altitude of the Sun with the help of the Sun's declination and the latitude of the place ; *BrSpSi*. III. 47.

Also *MBh*. III. 11 ; *LBh*. III. 27 ; *ŚiDVr*. I. iii. 16 ;*SiŚe*. IV. 42.

15. Rule for determination of the latitude with the help of the Sun's meridian zenith distance and declination :

BrSpSi. : III. 13.

Also *MBh*. III. 17; *LBh*. III. 34; *SūSi*. III. 15-16;*SiŚe*. IV. 51,

16 Rule for finding out the Rsine of the Sun's altitude for the given time in *ghaṭis* · *BrSpSi*. III. 25-26 Also *Ārya* IV. 28, *MBh*. III. 18-20, *LBh* III 7-10: *ŚiDVr*. I. iii. 24-25; *SiŚe*. IV. 32,34. *SiŚi*. I. iii. 53-54.

17. Rule for finding out the Sun's altitude : RSin $\alpha=$

$$\frac{M\times \text{day radius}}{R}\times\frac{\text{gnomon}}{\text{hypotenuse of equinoctial midday shadow}}$$

where M=R Sin (given *ghaṭis*∓asc diff) Rsin (asc. diff.), the upper or lower sign being taken according as the Sun is the northern or southern hemisphere. α is the Sun's altitude. *BrSpSi*. III. 27 (i).

Also *MBh* III 24 *ŚiDVr* I iii 27 *SiŚe* IV 37

18 Rule for finding the Sun's altitude when the Sun's ascensional difference is greater than the given time *BrSpSi* III 33

Also *MBh* III 25, *LBh* III 11 *ŚiDVr* I iii 29 *SiŚe* IV 41

19 Rule for finding the Sun's altitude in the night *BrSpSi* III 63

Also *MBh* III 26 *LBh* III 11 *SiŚe* IV 89

The Sun's altitude for the night has been called *Pātāla Śaṅku* by Brahmagupta (*BrSpSi* XV 9)

20 Rule for finding the longitude of the rising point of the ecliptic with the help of (i) the instantaneous sāyana longitude of the Sun and (ii) the civil time measured since sunrise or with the help of (i) the Sun's sāyana longitude at sunrise and (ii) the sidereal time elapsed since sunrise *BrSpSi* III 18 20

Also *MBh* III 30-32 *LBh* III 17 19 *SaSi* III 46-48
ŚiDVr I i i 11 12 *SiŚe* IV 18-19 (i) *SiŚi* I iii 2-4

21 Rule for obtaining the civil time measured since sunrise with the help of (i) the Sun's instantaneous sāyana longitude and (ii) the sāyana longitude of the rising point of the ecliptic or the sidereal time elapsed since sunrise with the help of (i) the Sun's sāyana longitude at sunrise and (ii) the sāyana longitude of the rising point of the ecliptic *BrSpSi* III 21 23

Also *SaSi* III 50-51 *MBh* III 34 36, *LBh* III 20,
ŚiDVr I iii 13 *SiŚe* IV 19 (ii)—22 (i) *SiŚi* I iii 5-7 (i)

22 Rule for determining the R Sines of the Sun's prime vertical altitude *BrSpSi* III 52.

Also *Ārya* IV *MBh* III 37 38, *LBh* III 52.

(An error created by Āryabhaṭa has been criticised by Brahmagupta)

23 Construction of the locus of the end of the shadow of a gnomon *BrSpSi* III 2-3

Also *MBh* III 52 *ŚiDVr* I iii 3 *SiŚe* IV 5

24 Rule for finding the Sun's mean anomaly *BrSpSi* II 12 (i)

Also *MBh.* IV. 1; *SuSi.* II. 29 ; *ŚiDVr* I. ii. 10; *SiŚe.* iii. 12; *SiŚi.* I. ii. 18-19 (i).

25. Rule for finding the RSine (Reversed sine) of an arc (<90°) : *BrSpSi* II. 10.

Also *SuSi.* II. 31-32 ; *MBh.* IV. 3-14; *LBh.* II. 2 (ii)-3 (i); *ŚiDVr.* I. ii. 12 ; *SiŚe.* III. 15; *SiŚi.* I. ii. 10 (ii)-11.

(We shall discuss it separately in the light of Brahmagupta formula.)

26. Rule for finding the Sun's equation of the centre: *BrSpSi.*. II. 15 (ii):

Also *MBh.* IV. 4 (ii): . III *SiŚe.* 27

27. Rule for determining the Sun's true longitude: *BrSpSi.* XIV. 17-18.

Also *MBh.* IV. 21-23; *SiŚe.* III. 52.

28. Rule for finding the Sun's *bhujāntara* correction under the eccentric theory : *BrSpSi.* XIV. 19.

Also *MBh.* IV. 24.

29. Rule for determining the *cara-saṁskāra* or *cara* correction : *KK.* I. 22.

30. Rule for finding the semi-durations of the day and night : *BrSpSi.* II. 60 ; *KK.* I. 23.

Also *SuSi.* II. 62-63 ; *ŚiDVr.* I. ii. 20-21 ; *SiŚe.* III. 70 ; *SiŚi.* I. ii. 52.

31. Rule for calculating the *tithi* : *BrSpSi.* II. 62 ; *KK.* I. 25.

Also *SuSi.* II. 66 ; *ŚiDVr* I. ii. 22 ; *SiŚe.* III. 71 ; *SiŚi.* I. ii. 66.

32. Rule for calculating the *Karaṇa* : *KK.* I. 27.

Also. *ŚiDVr.* I. ii. 24 , *SiŚe.* III. 77 ; *SiŚi.* I. ii. 66.

33. Rule for calculating *nakṣatra* : *BrSpSi.* II. 62 ; *KK.* I. 24.

Also *SuSi.* II. 64 ; *ŚiDVr.* I. ii. 23 (i) ; *SiŚe.* III. 75; *SiŚi.* I. ii. 67.

34. Rule pertaining to direct and retrograde motions of a planet : *BrSpSi.* II. 50-51.

Also *MBh* IV 56-57 *SiŚe* III 59 *ŚiDVr* I ii 42

35 A rule for converting true distances known in minutes into true distances into yojanas for example Sun s true distance in yojanas

$$\frac{\text{Sun s mean distance in } yojanas \times \text{Sun s true dist in minutes}}{\text{Radius}}$$

BrSpSi XXI 31 (ii)

Also *MBh* V 3 *ŚiDVr* I iv 5 (i) *LBh* IV 3 *SiŚe* V 4 (ii) *SiŚi* I v 5 (i)

36 Rule for finding angular diameters of the Sun and the Moon *BrSpSi* XXI 34 (ii)

Also *MBh* V 5 *ŚiDVr* I iv 8 *SiŚe* V 6 *SiŚi* I v 7

37 Formulae for the true (i e angular) diameters of the Sun the Moon and the shadow in terms of the true daily motions of the Sun and the Moon (Here by shadow is meant the section of the cone of the Earth s shadow at the Moon's distance) *BrSpSi* IV 6 (i) *KK* IV 2 (i)

Also *MBh* V 6 7 *ŚiDVr* I iv 9 *MSi* V 5 (ii)
SiŚe V 9 *SiŚi* I v 8-9

38 Rule for finding the *spaṣṭa valana* (resultant *valana*) for the circle drawn with half the sum of the diameters of the eclipsed and eclipsing bodies as radius *BrSpSi* IV 18 (i)

Also *MBh* V 46 *ŚiDVr* I iv 26

39 Method for calculating the phase of the eclipse for the given time *BrSpSi* IV 11 12

Also *MBh* V 62-63 *ŚiDVr* I iv 19 20 *SiŚe* V 14

40 Rule for the determination of the diameter of the Earth s shadow shadow i e the diameter of the Section of the where the Moon crosses it *BrSpSi* XXIII 8 9

Also *MBh* V 71 73 *Ārya* IV 39-40 *ŚiDVr* I iv 6 (ii)-7

41 Process of successive approximations in connection with calculations of a lunar eclipse *BrSpSi* IV 8-9

Also *MBh* V 75-76 *LBh* IV 10-12 *ŚiDVr* I iv 14-16 *SiŚe* V 12 13 *SiŚi* I v 12-13

42. Rule relating to the visibility-correction known as *akṣa-dṛkkarma* : *BrSpSi*. VI. 4.

Also *MBh*. VI. 1-2 ; *ŚiDVṛ*. I. vii. 3 (ii) ; *MSi*. VII. 4 ; *SiŚe*. IX. 7.

43. Rule relating to the visibility correction known as *ayanadṛkkarma* : *BrSpSi*. VI. 3 ; X. 17.

Slightly modified in *MBh*. VI. 2 (ii)-3 ; *ŚiDVṛ*. I. vii. 2-3 (i) · *SiŚe*. IX. 4-5 ; similar in *MSi*. VII. 2-3 ; more accurate in *SiŚi*. I. viii. 4-5.

44. Rule relating to the visibility of moon : *BrSpSi*. VI. 6 ; X. 32.

Also *MBh*; VI. 4-5 (i) *PSi*. V. 3 ; *ŚiDVṛ*. I. vii. 5 ; *SiŚe*. IX. 8 (i). 13.

45. Rule for calculating the phase of the Moon : *BrSpSi*. VII. 11 (ii)-12.

Also *MBh*. VI. 5 (ii)-7 ; *ŚiDVṛ*. I. ix. 12.

46. Rule for the determination of the Moon's true declination (i.e. the declination of the centre of the Moon's disc) : *BrSpSi*. VII. 5.

Also *MBh*. VI. 8 ; *ŚiDVṛ*. 1. viii 2 ; *SiŚe*. X. 7. (these are approximate rules ; a more accurate rule occurs in *SiŚi*. I. vii. 3 and 13).

47. Graphical representation of the elevation of the lunar horns in the first quarter of the month at sunset : *BrSpSi*. VII. 7-10.

Also *MBh*. VI. 13-17 ; *ŚiDV*. I. ix ; *SiŚi*. I. ix.

48 Minimum distances of the planets from the Sun when they are visible : *BrSpSi*. vi 6 ; X. 32.

Also *MBh*. VI. 44 ; *ŚiDVṛ*. I. vii. 5 (i) ; *SiŚe*. IX. 8 (i). 12.

49. Rule relating to the determination of the time and the common longitude of two planets when they are in conjunction in longitude : *BrSpSi*. IX. 5-6.

Also *MBh*. VI. 49-51 ; *ŚiDVṛ*. I. x. 7-9 (i) ; *SiŚe*. XI. 12-12

50. Rule relating to the distance between two planets

which are in conjunction in longitude B_rSpS_1 IX 11

Also *MBh* VI 54 *ŚiDVr* I x 11 *SiŚe* XI 10

51 Rule For finding the *Bhujaphala* and *Koṭiphala* etc without the use of the RSine-difference table $BrSpS_1$ XIV 23 24

Also *MBh* VII 17 19 *SiŚe* III 17

52 To obtain the Sun s mean true longitude derived from the midday shadow of the gnomon $BrSpS_1$ XIV 28 III 61-62

Also *MBh* VIII 5 *SiŚi* I ii 45

53. *Rule to find the arc corresponding to a given RSine* $BrSpS_1$ II 11

Also *MBh* VIII 6 *SaSi* II 33 *ŚiDVr* I ii 13 *SiŚe* III 16 *SiŚi* I ii 11 (ii)—12 (i)

For the concordance given here we express our indebtedness to the work of K S Shukla on the *Mahābhāskarīya*

Tables of Constants

Some of the Tables of Constants have been given in an earlier chapter We here give a few more tables which would indicate how far Brahmagupta introduced new concepts in evaluating these constants of greater accuracy and refinement.

TABLE I

Position of Planets for the Beginning of *kaliyuga*

In this Table are given the positions of the planets including the Moon s apogee and ascending node for the beginning of *Kaliyuga* The calculations of Brahmagupta are different from those of the *Sūryasiddhānta* and of Āryabhaṭa I

Planet	Positions $BrSpS_1$	according *Ārya*	to $SiiS_1$
1	2	3	4
	s ° ′ ″	s ° ′ ″	s ° ′ ″
Sun	0 0 0 0	0 0 0 0	0 0 0 0
Moon	0 0 0 0	0 0 0 0	0 0 0 0

1	2	3	4
	s ° ′ ″	s ° ′ ″	s ° ′ ″
Moon's apogee	4 5 29 46	3 0 0 0	3 0 0 0
Moon s asc node	5 3 12 58	6 0 0 0	6 0 0 0
Mars	11 29 3 50	0 0 0 0	6 0 0 0
Mercury	11 27 24 29	0 0 0 0	0 0 0 0
Jupiter	11 29 27 36	0 0 0 0	0 0 0 0
Venus	11 28 42 14	0 0 0 0	0 0 0 0
Saturn	11 28 46 34	0 0 0 0	0 0 0 0

TABLE II

Diameters of the Sun the Moon and the Earth in *yojanas* and the distances of the Sun and the Moon from the Earth

	BrSpSi	Bhāskara I	Śrīpati	Bhāskara II	Modern (in miles)
1	2	3	4	5	6
Sun s diameter in *yojanas*	6,522	4,410	6 522	6,522	86 400
Sun s distance in *yojanas* (mid night reck)	689 358	459,585	684 870	689,377	92,900 000
Ratio		0 009596	0 009,523	009 461	00 093
Moon's diameter in *yojanas*	460	315	480	400	2,160
Moon s distance in *yojanas* (mid night reck)	51,566	34 377	51 566	51 566	2,389 000
Ratio Earth s dia meter in *yojanas*	1,581	0 009 163	0.009,308	009 308	0 009

TABLE III

Sidereal Revolutions of the Apogees of the Planets in a Kalpa

Apogee of	*BrSpSi*	According to *Sūrya Siddhānta*	*Āryabhaṭīya*
1	2	3	4
Sun	480	387	not given
Mars	292	204	
Mercury	332	368	
Jupiter	855	900	
Venus	653	535	
Saturn	41	39	

TABLE IV

Sidereal Revolutions of the Nodes of the Planets in a Kalpa

(Not given in the *Āryabhaṭīya*)		*Sūrya Siddhānta*
Node of	*BrSpSi*	
1	2	3
Mars	267	214
Mercury	521	488
Jupiter	63	174
Venus	893	903
Saturn	584	662

TABLE V

Peripheries of the Epicycles of the Planets

Planet	*BrSpSi* odd quad	*BrSpSi* even quad	*Sūrya Siddhānta* odd quad	*Sūrya Siddhānta* even quad	*Āryabhaṭīya* odd quad	*Āryabhaṭīya* even quad
1	2		3		4	

(a) *Manda* epicycles

Planet	*BrSpSi* odd quad	*BrSpSi* even quad	*Sūrya Siddhānta* odd quad	*Sūrya Siddhānta* even quad	*Āryabhaṭīya* odd quad	*Āryabhaṭīya* even quad
Sun	13°40′		13°40′	14°	13°30′	
Moon	31°36′		31°40′	32°	31°30	
Mars	70°		72°	75°	63°	81°
Mercury	38°		28°	30°	31°30′	22°30′
Jupiter	33°		32°	33°	31°30′	36°
Venus	9°	11°	11°	12°	18°	9°
Saturn	30°		48°	49°	40°30	58°30

(b) *Śīghra* epicycles

1	2		3		4	
Mars	243°40′[1]		232°	235°	238°30	229°30′
Mercury	132°		132°	133°	139°30′	130°30′
Jupiter	68°		72°	70°	72°	67°30
Venus	263°	258°	260°	262°	265°30′	256°30′
Saturn	35°		40°	39°	40°30′	36°

1 In the middle of quadrants it is 237°

TABLE VI

Mean Diameters of the Planets

Planet	*BrSpSi*	*Surya-Siddhanta*	*Aryabhatiya*	Modern
1	2	3	4	5
Sun	32′31″	32′24″	33′ approx	32′2 36″
Moon	32′ 1″ approx.	32′	31′30″	31′8″
Mars	4′46″	2′	1′17″	9′36″
Mercury	6′14″	3′	2′8″	6′68″
Jupiter	7′22″	3′30″	3′12″	3′14 72″
Venus	9′	4′	6′24″	16 8″
Saturn	5′24″	2 30″	1′36″	2′49.5″

TABLE VII

Inclination of the Orbits of the Planets to the Ecliptic

Planet	*BrSpSi*	*Surya-Siddhanta*	*Aryabhatiya*	Modern (Jan 00,195°)
1	2	3	4	5
Moon	4°30′	4°30′	4°30′	5°8′40″
Mars	1°50′	1°30′	1°30′	1°51′0″
Mercury	2°32′	2°	2°	7°0′14″
Jupiter	1°16′	1°	1°	1°18′21″
Venus	2°16′	2°	2°	2°23′39″
Saturn	2°10′	2°	2°	2°29′25″

TABLE VIII

Longitudes of the Junction Stars according to Different Authorities

Junction-star of	Longitude (polar) according to		Longitude according to		
	BrSpSi KK. SiŚi.	*SiŚe SaSi*	*MBh.*	*LBh.*	*SiDVr.*
1	2	3	4	5	6
Aśvinī	8°	8°	8°	8°	8°
Bharanī	20°	20°	27°	26°30°	20°
Kṛttikā	37°28′	37°30′	36°	36°	36°
Rohiṇī	49°28′	49°30′	49°	50°	49°
Mṛgaśirā	63°	63°	62°	62°	62°
Ārdrā	67°	67°20′	70°	70°	70°
Punarvasu	93°	93°	92°	92°	92°
Puṣya	106°	106°	105°	105°	105°
Āślesā	108°	109°	114°	114°	114°
Maghā	129°	129°	128°30′	128°30′	128°
P-Phālgunī	147°	144°	141°	141°	139°20′
U-Phālgunī	155°	155°	154°	154°	154°
Hasta	170°	170°	173°	173°	173°
Citrā	183°	180°	185°	185°	184°20′
Svāti	199°	199°	197°	197°	197°
Viśākhā	212°5′	213°	212°	212°	212°
Anurādhā	224°5′	224°	222°	222°	222°
Jyeṣṭhā	229°5′	229°	228°	228°	228°
Mūla	241°	241°	241°	241°30′	241°
P-Āṣāḍhā	254°	254°	254°	254°30′	254°

1	2	3	4	5	6
U-Āsāḍha	260°	260°	267°	266°30′	267°20′
Śravana	278°	280°	285°	284°30′	284°10′
Dhaniṣṭhā	290°	290°	295°	295°30′	295°20′
Śatabhiṣak	320°	320°	307°	307°	313°20′
P-Bhādra	326°	326°	323°	328°	327°
U Bhādra	337°	337°	345°	345°	335°20′
Revatī	0°	359°50′	360°	360°	359°

TABLE IX

Celestial Latitudes of the Junction-*Stars*

Junction star of	Polar latitude given in			Latitude given in		
	BrSpSi *SiŚe*	*KK*, *SaSi*	*SiŚi*	*MBh*	*LBh*	*ŚiDVṛ*
1	2	3	4	5	6	7
Aśvinī	10°*N*	10°*N*	10°*N*	10°*N*	10°*N*	10°*N*
Bharaṇī	12°*N*	12°*N*	12°*N*	12°*N*	12°*N*	12°*N*
Kṛttikā	4°31′*N*	5°*N*	4°30′*N*	5°*N*	5°*N*	5°*N*
Rohiṇī	4°33′*S*	5°*S*	4°30′*S*	5°*S*	5°*S*	5°*S*
Mṛgaśiri	10°*S*	10°*S*	10°*S*	10°*S*	10°*S*	10°*S*
Ārdrā	9°*S*	9°*S*	9°*S*	9°*S*	9°*S*	9°*S*
Punarvasu	6°*N*	6°*N*	6°*N*	6°*N*	6 *N*	6°*N*
Puṣya	0	0	0	0	0	0
Āśleṣā	7°*S*	7°*S*	7°*S*	7°*S*	7°*S*	7°*S*
Maghā	0	0	0	0	0	0
P-Phālgunī	12 *N*	12°*N*	12°*N*	12°*N*	12°*N*	12°*N*
U Phālgunī	13°*N*	13°*N*	13°*N*	13°*N*	13°*N*	13°*N*

1	2	3	4	5	6	7
Hasta	11°S	11°S	11°S	7°S	7°S*	8°S
Citrā	1°45′S	2°S	1°45′S	2°S	2°S	2°S
Svāti	37°N	37°N	37°N	37°N	37°N	37°N
Viśākhā	1°23′S	1°30′S	1°20S	1°30′S	1°30′S	1°30′S
Anurādhā	1°44′S	3°S	1°45′S	3°S	3°S	3°S
Jyeṣṭhā	3°30′S	4°S	3°30′S	4°S	4°S	4°S
Mula	8°30′S	9°S	8°30′S	8°20′S	8°0′S	8°30′S
P-Āṣāḍhā	5°20′S	5°30′S	5°20′S	7°S	7°S	5°20′S
U-Āṣāḍhā	5°S	5°S	5°S	5°S	5°S	5°S
Śravaṇa	30°N	30°N	30°N	30°N	30°N	30°N
Dhaniṣṭhā	36°N	36°N	36°N	36°N	36°N	36°N
Śatabhisak	18′S	30′S	20′S	18′S	18′S	20′S
P-Bhādra	24°N	24°N	24°N	24°N	24°N	24°N
U-Bhādra	26°N	26°N	26°N	26°N	26°N	26°N
Revatī	0	0	0	0	0	0

—:o:—

Reference

K S. Shukla . The *Mahābhāskarīya* and the *Laghubhāskarīya*

CHAPTER XII

Brahmagupta
A Great Critic

The eleventh chapter of the *Brāhmasphuṭasiddhānta* is known as the *Tantra parīkṣādhyāya* literally meaning the critical examination of the existing systems in which Brahmagupta has attempted to deal with those essentials in which he differs from the then existing authorities. In this chapter he specially criticises Āryabhaṭa I the Jain systems the views of Śrīṣena Viṣṇucandra Pradyumna Lāṭasimha and foreign astronomers evidently Greek and others The chapter consists of sixty three verses, and is of historical significance Undoubtedly Brahmagupta is unsparingly critical and he condemns his opponent in the strongest possible words. He also criticises the Saṁhitākāras as Pitāmaha and others in their astronomical notions We shall try to give a brief summary of some of these views where Brahmagupta differs from other authorities

1 Brahmagupta does not accept a *yuga* of five years a view which is propounded by the authors of the Vedāṅga Jyautiṣa such as Lagadha and other authors as Pitāmaha [1] Varāhamihira regards one *adhimāsa* in a period of thirty solar months and here too Brahmagupta differs (latter authorities assign one adhimāsa in a period of 32 solar months and 16 days)

2. Brahmagupta does not accept the views of the Jainas that there exist two Suns and two Moons and fifty four nakṣatras. The reason advanced by Brahmagupta is that Bharani (or *Dhruvamatsya* returns to its original position (facing to the east

1 [illegible]
[illegible]
—*BrSpSi* XI 2

backing to the west) in one day It is sufficient to postulate the existence of only one Sun, one Moon and twenty-seven naksatras to explain the astronomical phenomena [1]

3 Brahmagupta differs from Āryabhaṭa I in the length of the four *yugas* Ārvabhaṭa regards all the four *yugas* of equal lengths, i e. 1,080,000 years, the *caturyuga* being of 4,320 000 years Brahmagupta regards *Kaliyuga* to be of 432,000 years, *Dvāpara* to be twice of it, *Tretā* to be thrice of it and *Kṛtayuga* of four times of the length of the *Kaliyuga* Both have the *caturyuga* of the same length [2]

4 Āryabhaṭa was not clear with respect to the number of civil days (*sāvana dina*) in a *yuga* , in one of his treatises he gives this number to be 1,577,917 800 and in the other 1,577 917, 500 with a difference of 300 days though in both the treatises Āryabhaṭa I regards the number of solar years to be 4,320,000 in a *Caturyuga* or *Mahāyuga* Why is this difference ? asks Brahmagupta [3]

5 Āryabhaṭa regards *mandocca* (the apogee) and *pāta* (the ascending node of the orbit on the ecliptic) as constant or stationary, then how could he propound a *sphuṭa-yuga* or the concept of true *yuga* with the concurrence of year, month and day on the Caitra Śukla Pratipadā (the first day of bright half of the month Caitra) at the same time as indicated by Āryabhata in his *Laghvāryabhaṭīya Tantra* [4]

Āryabhaṭa was not clear in respect to the variance in the pāta In the Āryāṣṭa sata (in the Āryabhaṭīva which has 108 Ārya verses), Āryabhaṭa states that the *pāta* of all the planets

1 भानिचतुस्पञ्चाशद्द्वौ द्वावर्कौ द्वौ जिनोक्तं यत् ।
भ्रमकालस्यासत्त्वात् भवति यतोऽन्द्या तत्तदसत् ॥ BrSpSi XI 3

2 आर्यभटोयुगपादांस्त्रीन् यातानाह कलियुगादौ यत् ।
तस्यकृतान्तस्मात् स्वयुगाद्यन्तौ न तत् तस्मात् ॥ —BrSpSi XI 4

3 युगरविभगणाः ख्युघृ इति यत्प्रोक्तं तत् तयोर्युगं स्पष्टम् ।
त्रिशती ख्युदयानां तदन्तरं हेतुना केन ॥ —BrSpSi XI 5

4 युगपादान् वदताचैत्रसितादेः क्रम प्रवृत्तान् यत् ।
तदसत् यत स्फुटयुग तत् स्थैर्यान्मन्दपातानाम् ॥ —BrSpSi XI 6

Mańgalavāra (Tuesday), (*Budhavāra*) (Wednesday), *Guruvāra* (Thursday) Sukravāra (Friday) and Śanivāra (Saturday) Thus Āryabhaṭa gives the same order of days as other authorities and there is no reason why he should be criticised

Certainly what is Sunday for Lankā may not be Sunday at the same time for Siddhapura and Āryabhaṭa should have emphasised that the day (Monday Tuesday etc) is not constant for all the places [1]

10 Brahmagupta expresses surprise why Āryabhaṭa, in two of his treatises propounds two different systems of reckoning one from the Sunrise in Lankā and the other from the Midnight in Lankā [2] This causes a difference of one-fourth of the daily-motion in the two reckonings of the motion of planets [3]

11 Brahmagupta criticises Āryabhaṭa on the point of diameter of the Earth In the *Gītikāpāda* 5 and 6, Āryabhata states that one yojana = 8 000 × puruṣa, and 1 puruṣa = 4 hasta, thus 1 yojana = 32 000 purusas and the diameter of the Earth is 1050 yojanas Brahmagupta further says that an error in the diameter of the Earth would cause an error in deśāntara or longitude and thus also in the true *tithi* and consequently in the calculation of eclipses also [4]

12 Āryabhata has rightly stated that the Earth is in motion and the *Bhagaṇas* are stationary Brahmagupta's objection is that if the Earth is in motion, birds would not be able to return to their nests and if the Earth's motion is upside-down

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1 सूर्यादयश्चतुर्थो दिनवारा यदुवाच तदसदार्यभट: ।
लङ्कोदयो यतो ऽर्कस्यास्तमय प्राह सिद्धपुरे ॥ —*BrSpSi* XI 12

2 अधिकै रात्रैश्चतुर्भिर्वर्षसहस्रैश्चतुर्दशाभिरेक ।
युगपादे दिनवारास्तैरौदयिकार्धरात्रिकयो ॥ —*BrSpSi* XI 13

3 औदयिकादिनभुक्तेश्चतुर्थांशो नार्धरात्रिको भवत्यून ।
कतर स्फुट न निश्चितमनयो स्फुटमेकमपि नात ॥ —*BrSpSi* XI 14

4 षोडशभागवियोजनरहि प्रतिभूव्यास पुलावदता ।
आत्मज्ञान तस्यापि त्वनिश्चयस्तानि कृतकत्वात् ॥
भूव्यासस्याज्ञानाद् व्यर्थं देशान्तरं तदज्ञानात् ।
स्फुटतिथ्यज्ञानात् तिथिनाशाद् ग्रहणयोर्नाश ॥ —*BrSpSi* XI 15-16

then the roof and hills would come down which is contrary to our observation[1] Obviously Brahmagupta is not ustified in his criticism

13 Brahmagupta points out to the differences in his calculations and the calculations of Āryabhaṭa in the peripheries of the *manda* and *śīghra* epicycles of planets in the odd and even quadrants (This difference we have shown in Table V p 313)[2]

14 Āryabhaṭa I and Bhāskara I have both given a rule for the determination of the *dṛkkṣepajyās* of the Sun and the Moon

> Take the product of the Sun s or Moon s own *madhyajyā* and *udayajyā* then divide the product by the radius and then take the square of the quotient Subtract that from the square of the own *madhyajyā* the square root of that difference is known as the Sun s or Moon s *dṛkkṣepajyā*[3]

The Sun s *dṛkkṣepajyā* is the Rsine of the zenith distance of that point of the ecliptic which is at the shortest distance from the zenith (*this point is called nonagesimal or the central ecliptic point*) The Moon s dṛkkṣepajyā is the Rsine of the zenith distance of that point of the Moon s orbit which is at the shortest distance from the zenith The rule given above is only approximate and has been criticised by Brahmagupta[4]

15 In the *Āryabhaṭīya* there is a rule in the Golapāda for finding the Rsine of the *agrā* of the true Sun and also for the

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1 [illegible]
[illegible] —*BrSpSi* XI 17

2 [illegible]
[illegible]
[illegible]
[illegible] —*BrSpSi* XI 18-19

3 [illegible]
[illegible] —*Mbh* V 19

4 [illegible]
[illegible]
[illegible]
[illegible] —*BrSpSi* XI 29 30

Rsine of the Sun's prime vertical altitude.[1] Bhāskara also gives the rule in his *Mahābhāskarīya* :

> Multiply the Rsine of the (Sun's) greatest declination by the Rsine of the Sun's true (*sāyana*) longitude; then divide (the product) by the Rsine of the colatitude, the result is (the Rsine of) the *agra* of the true Sun. When that (*agra*) is less than the latitude and when the Sun is also in the northern hemisphere, multiply (the Rsine of the Sun's *agra*) by (the Rsine of) the colatitude; the result is the Rsine of the Sun's prime vertical altitude.[2]

The condition laid down in the rule that the "Sun's *agra* should be less than the latitude" is incorrect. The error was originally comitted by Āryabhata and Bhāskara followed it. This error was noticed by Brahmagupta[3]. Bhāskara I, however, corrected the error in the *Laghu-Bhāskarīya*. There he gives the correct conditions It is not the *agra* that should be less than the latitude, it is the Sun's declination (or rather the Rsine of the Sun's northern declination as we have in the *Laghu-Bhāskarīya*), which should be less than the latitude (or rather the Rsine of the latitude). This condition is necessary for the existence of the prime vertical shadow of the gnomon.

It may be pointed out that the commentators of Āryabhata I have interpreted the rule given by Āryabhaṭa I as conveying the correct meaning : they say that Āryabhaṭa also meant declination when he used the term *agra*.

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1. परमापक्रमजीवामिष्टज्यार्धाहतां ततो विभजेत् ।
ज्यालम्बकेन लब्धार्काग्रा पूर्वापरे क्षितिजे ॥
सा विपुवज्ज्योना चेद् विपुवदुदग्लम्बकेन संगुणिता ।
विपुवज्ज्यया विभक्ता लब्धः पूर्वापरे शङ्कुः ॥ —*Ārya*. V. 30-31

2. स्फुटरविभुजनिघ्ना या परा क्रान्तिजीवा ।
हरतु समवलम्बज्या कलापेन भूयः ॥
स्फुट दिवस कराग्रा सा यदाऽक्षांशहीना ।
रविरपि यदि गोले चोत्तरे लम्बकघ्नाम् ॥
अक्षज्यया हरेद् भूयः शंकुः स्यात् सममण्डले ।
तद् वर्गं व्यास कृत्योर्यद् विश्लेषं तत्पद प्रभा ॥ —*MBh*. III. 37-38

3. उत्तरगोलेऽग्रायां विपुवज्ज्यातो यदुत्तमूनायाम् ।
सममण्डलगत्वमतः क्रान्तिज्याया यतो भवति ॥ —*BrSpSi*. XI. 22

Usually by *agra* we mean the arc of the celestial horizon lying between the east point and the point where a heavenly body rises or between the west point and the point where a heavenly body sets Declination is *krānti*

16 Brahmagupta has criticised Āryabhaṭa and his group for their expressions for determining *lambana* (i e the difference of the parallaxes in longitude of the Sun and the Moon) and the rule for determining the *avanati* or *nati* (i e the difference in parallaxes in latitude of the Sun and Moon)[1]

Lambana is obtained with the help of the five Rsines (i) *madhya jyā* (ii) *udaya-jyā* (iii) *dṛk kṣepa jyā* (iv) *dṛg jyā* and (v) *dṛg-gati jyā*

(i) The *madhyajyā* is the Rsine of the zenith distance of meridian ecliptic point

madhyajyā=Rsin ($\phi \pm$ declination of the meridian ecliptic point)[2]

In this expression ϕ is the latitude of the place and by R sin is meant R$\times$sine R being the radius of the celestial sphere

(ii) The *udayajya* is the Rsine of the arc of the horizon intervening between the equator and the ecliptic and is given by

$$udayajya = \frac{R \sin L \times R \sin \varepsilon}{R \cos \phi}$$

where L is the longitude of the horizon ecliptic point in the east and ε the obliquity of the ecliptic

(iii) The *dṛkksepajyā* is the R sine of the zenith distance of the central ecliptic point[3] and is given by

1 व्यासार्धेन विभक्ता दृग्गतिजीवा चतुर्गुणा लम्बनम् ।
लम्बननाडय पञ्चदशगुणितया त्रिज्यया भक्ता ॥
दृक्क्षेपज्या गुणयन्त्याद्यना लम्बनवन्निर्भवति ।
स्फुटयोजनकर्णाभ्यां भूव्यासेन च विना स्पष्टे ॥
आर्यभटेनास्मिन् सति लघुनि विमर्शं स्पष्टं कृतं कर्म ।
गणितं जानाज्ज्यार्थं विज्ञानता यदि सत स्फुटाम् ॥ —*BrSpSi* XI 23-25

2. The meridian ecliptic point is the point of the ecliptic on the meridian,

3 The central ecliptic point is the central point of the portion of the ecliptic lying above the horizon.

dṛkkṣepajya

$$=\left[(madhyajyā)^2-\left\{\frac{udayajyā \times madhyajyā}{R}\right\}^2\right]^{\frac{1}{2}}$$

where R is the radius of the celestial sphere.

(iv) The *dṛgjyā* is the Rsine of the zenith distance (of the Sun) and is given by :

$$dṛgjyā=\left[R^2-\left\{\frac{dṛggatijyā \times R \sin(L-\theta)}{R}\right\}\right]^{\frac{1}{2}}$$

where L is the longitude of the horizon ecliptic in the east and θ the longitude of the Sun.

(v) The *dṛggatijya* is the Rsine of the altitude of the central ecliptic point, and is given by :

$$dṛggatijyā=[R^2-(dṛkkṣepajyā)^2]^{1/2}$$

where R is the radius of the celestial sphere.

In the *Mahābhāskarīya*[1], the expression for the Sun's *dṛggatijyā* is

$$(\text{Sun's } dṛggatijyā)^2$$
$$=(\text{Sun's } dṛgjyā)^2-(\text{Sun's } dṛkkṣepajyā)^2$$

and similar is the expression for the the Moon's *dṛggatjya*.

Now *lambana*, which is the difference of the parallaxes, in longitude, of the Sun and the Moon, is given by the expression :

$$Lambana = \text{Moon's } lambana - \text{Sun's } lambana.$$

Sun's *lambana*

$$=\frac{\text{Sun's } dṛggatijya \times \text{Earth's semidiameter}}{\text{Sun's true distance in yojanas}}$$

Moon's *lambana*

$$=\frac{\text{Moon's } dṛggatijya \times \text{Earth's semidiameter}}{\text{Moon's true distance in yojanas}}$$

These *lambanas* are in terms of minutes of arc etc[2]

1 [illegible]
[illegible] —*MBh* V, 23

2. [illegible]
[illegible]

(Cont. on Page 225)

Thus *lambana* is given by subtracting the Sun s *lambana* from Moon s *lambana*

This *lambana* is also expressed in the following way

Lambana

$$= \left[\frac{\{(drgjy\bar{a})^2-(drkk\d{s}epajy\bar{a})^2\}^{1/2}\times 18}{\text{Moon s true distance}} - \frac{\{(drgjy\bar{a})^2-(drkk\d{s}epajy\bar{a})^2\}^{1/2}\times 18}{\text{Sun s true distance}} \right]$$

in minutes

$= \frac{60}{d} \times$ (*lambana* calculated in minutes) is the *lambana* in *ghaṭis* where d denotes the difference between the daily motions of the Sun and the Moon

Āryabhaṭa I has given his description of the determination of *lambana* and *avanati* in the Golapāda 33 34 of the *Āryabhaṭīya*[1] and Bhāskara has followed his rules in the *Mahābhaskarīya*[2] Brahmagupta criticises them in his *Brāhma sphuṭasiddhānta*[3]

(vii) We shall now take up *nati* or *avanati* (both the terms mean the same) *Nati* is the difference of the parallaxes in *latitude* of the Sun and the Moon and is given by

$$nati = \left[\frac{drkk\d{s}epajy\bar{a}\times 18}{\text{Moon s true dist}} - \frac{drkk\d{s}epajy\bar{a}\times 18}{\text{Sun s true dist}} \right]$$

minutes.

(Con from Page 324)

तद्विशेषो हत षष्ट्या स्फुटभुक्त्यन्तरोद्धृत ।
घटिकादिस्तिथे ग्राह्या शुद्धि च पोऽपरे मत ॥
ग्नारकालनिष्पन्न लम्बन शोध्यते तिथे ।
दृग्गिन्दृग्ज्याया दृश्यते तत्र दक्षिणे ॥
एव पुन पुन कर्म यावत्तन्निर्विशिष्यते ।
तिथिवच्च दृतीकृणाश् स चार्यावेव पठ्यन्ते ॥ —*MBh* V 24 27

1 मध्यज्योदयजीवासंवर्गो व्यासदलहृते यत् स्यात् ।
तन्मध्यज्याकृत्योर्विशेषमूलं स्वदृक् क्षेप ॥
दृग्गोलकृति विशेषित्मूलं स्वदृग्गति कुदरात् ।
क्षितिजे स्व दृक छाया भूव्यासार्द्धं नभोमध्यात् ॥ —*Ārya.* IV 33 34

2 loc. cit. *MBh* V 24 27

3 loc cit *BrSpSi.* XI 23-25

(viii) Moon's true latitude = Moon's latitude × *nati*

The present Sūryasiddhānta and Brahmagupta both utilise the following expressions for *lambana* and *nati* which give more accurate values

$$lambana = \frac{\text{Rsin}\,(M-\odot) \times drggatijyā}{\{\text{Rsin}\,(30^\circ)\}^2} \text{ ghaṭis}$$

where M and $\odot$ denote the longitudes of the meridian ecliptic and the Sun respectively

$$nati = \frac{dṛkkṣepajyā \times d}{15 \times R}$$

where R is the radius of the celestial sphere and d denotes the difference between the daily motions of the Sun and the Moon[1]

Brahmagupta has raised objections to the Āryabhaṭa system regarding *lambana* (XI 26 28) *dṛkkṣepa* (XI 30 31) *ayanadṛkkarma* (XI 35) elevation of Moon's horns (*śṛṅgonnati*) (XI 39) and similar several other points He is so vehemently opposed to Āryabhaṭa that finally he declares

> "It is beyond my capacity to enumerate all the defects of Āryabhaṭa Only a few have been given here as illustration Intelligent people can easily find out others[2]

He also says

> Āryabhaṭa is neither conversant with the Gaṇita (mathematics) nor Kāla (time calculations) nor Gola (celestial or spherical calculations) It is difficult to enumerate separately the fallacies committed by him in the respective chapters of the *Gaṇitapāda* *Kālakriyāpāda* and *Golapāda*[3]

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1 loc cit *BrSpS* XI. 23

2 आर्यभटदूषणानां संख्या वक्तुं न शक्यते यस्मात् ।
तस्मादयमुद्देशो बुद्धिमतान्यानि योज्यानि ॥ —*BrSpSi* XI 44

3 जानात्येकमपि यतो नार्यभटो गणितकालगोलानाम् ।
न मया प्रोक्तानि ततः पृथक् पृथग् दूषणान्येषाम् ॥ —*BrSpSi* XI 43

Brahmagupta and Śrīṣeṇa

In Varāhamihira s *Pañcasiddhāntikā* we have a critical review of the five *Siddhāntas* or five systems of astronomical study Puliśa Siddhānta Romaka Siddhānta Vasistha Siddhānta Sūrya Siddhānta and Brāhma Siddhānta Colebrooke in his Paper *On the notion of the Hindu Astronomers concerning the precession of the equinoxes and motions of the planets* publi shed in the *Asiatic Researches* vol xii p 209 250 Calcutta 1816 4 to reproduced in the *Miscellaneous Essays* Vol II 18 2 says the following in regards to the authorship of these schools of astronomy

> All these books are frequently cited in the astronomical compilations and are occasionally referred to their real or supposed authors The first is everywhere assigned to Puliśa whose name it bears The *Romaka Siddhānta* is ascribed by the scholiast of Brahmagupta and by a commentator of the *Sūrya Siddhānta* to Śrisena The Vāsistha Siddhānta is by the same authority given to Visnucandra Both these authors are repeatedly mentioned with censure by Brahma gupta and it is acknowledged that they are entitled to no particular deference
>
> The *Brāhma Siddhānta* which is the basis of Brahma gupta s work is not anywhere attributed to a known author but referred to in all quotations of it which have fallen under observation either to the *Viṣṇudharmottara Purāṇa* of which it is considered as forming a part or to Brahmā (also called Pitāmaha) who is introduced into it as the speaker in a dialogue with Bhrgu or it is acknowledged to be the work of some unknown person The true author it may be now impracticable to discover and would be vain to conjecture
>
> The *Sūrya Siddhānta* (if the same which we now possess) is in the like manner ascribed to no certain author unless in the passage cited by my colleague Mr Bently (*Asiatic Researches* vol vi p 572) who says that in the commentary of the *Bhāsvatī* it is declared that

> Varāha was the author of the *Sūrya Siddhānata*", and who adds that 'Satānanda, the author of the *Bhāsvati* was a pupil of Varāha under whose directions he himself acknowledges, he wrote that work'
>
> This concluding remark alludes to the following verse of the *Bhāsvati Karana* "Next I will propound succinctly, from Mihira's instruction, (the system) equal to the *Sūrya Siddhānta*[1], (*Miscellaneous Essays* p 388-90) (The word 'Mihira' has double meaning it might be an abbreviation of *Varāhamihira* or it may mean sun or *Sūrya*)

Thus on the authority of Colebrooke Srīsena may by regarded as the initiator of the Romaka system Brahmagupta him self mentions in one of his passages the name of Sriṣena in connection with the Romaka system and further the conceptions of the Romaka system came down as Vāsistha system through Visnucandra [2] Lātadeva also derived from Sriṣena the concepts of the mean motions of the Sun the Moon the Moon s apogee and her node and the mean motions of Mars Mercury s *Śighra* Jupiter, Venus' *sigrha* and Saturn I have indicated elsewhere, which is also the view of Sankara Bālakrsna Diksita that the original Romaka and Paulisa Siddhāntas were introduced to Indians by Lātadeva and the latter Romaka Siddhānta by Sriṣena (Original Romaka Siddhanta was prevalent before Śaka 427 and this is the one which is mentioned by Varāhamihira who makes no reference to Srīsena and Viṣnucandra in the *Pancasidhāntikā* and the latter Romaka Siddhānta was introduced by Śrisena as is indicated by Brahmagupta Thus we have two Vāsiṣṭha Siddhāntas and two Romaka Siddhāntas) My personal view is that Lāṭadeva Śrisena and possibly Visnucandra also, were naturalised Greeks settled in India and they had adopted themselve to Indian life They were conversant in Greek and Indian Astronomy both and had contributed substantially to Indian astronomy Brahmagupta was opposed to any of these foreign influences dominating Indian

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1 अथ प्रवक्ष्ये मिहिरोपदेशात् तत्सूर्यसिद्धान्तसमं समासात् ॥

—*Bhāsvati Karaṇa*

2 श्रीषेणेन गृहीत्वा रत्नोच्चयरोमकं कृतं कन्था ।
एतानेव गृहीत्वा वासिष्ठो विष्णुचन्द्रेण ॥

—*BrSpS*: XI 50

systems and he very much resented such interferences in pure academic life of this country He was opposed to Āryabhaṭa for a different reason Āryabhaṭa was universally regarded as an authority in this country and the conservativism was so deep that even where it could be shown by direct observation or on valid theoretical grounds that a particular concept was erroneous or less accurate people still chose to adhere to it since they had the backing of Āryabhaṭa s authority Brahmagupta was against this nonscientific attitude Needless to say Brahmagupta was not always fair to Aryabhaṭa in his criticism he overdid in enumerating the shortcomings of Āryabhaṭa s system as if he was personally jealous of his wide popularity

Brahmagupta s feelings against Lāṭadeva Śrīṣeṇa Viṣṇucandra and others would be seen from the following passage in the *Brāhmasphuṭasiddhānta*

[Cont on Page 33.]

Brahmagupta very emphatically says about his system that so long as people would be finding concordance between the observed and theoretical results *(drggamitaikyam)* in respect of solar and lunar eclipses his Brāhma Siddhānta would be held in esteem[2]

In other systems whatever concordance appears to be between the observation and calculation of eclipses etc it is, Brahmagupta says merely accidental or by chance, as the maxim of letters bored by an insect in wood or paper[3].

युक्त्याऽऽर्यभटोक्तानि प्रत्येकं दूषणानि योज्यानि ।
श्रीषेणप्रभृतीनां कानि चिह्नानि वक्ष्यामि ॥ [*Cont from Page 329*]

लाटात् सूर्यशशाङ्कौ मध्याविन्दूच्चं चन्द्रपातौ च ।
कुजबुधशीघ्रबृहस्पति सितशीघ्र शनैश्चरान् मध्यान् ॥

युगयातवर्षभगणान् वासिष्ठाद्विजयनन्दि कृतपादात् ।
मन्दोच्च परिधिपातस्पष्टीकरणाद्यमार्यभटात् ॥

श्रीषेणेन गृहीत्वा रत्नोच्चयरोमक कृत कन्था ।
एतानेव गृहीत्वा वासिष्ठो विष्णुचन्द्रेण ॥ —*BrSpSi* XII 46-50

2 चन्द्रार्कग्रहणेन्दुच्चाद्यादिषु सर्वदा यतो ब्राह्मे ।
दृग्गणितैक्यं भवति स्फुटसिद्धान्तस्ततो ब्राह्मः ॥ —*BrSpSi* XI 61

3 अनयोर्न कदाचिदपि ग्रहणादिषु भवति दृग्गणितैक्यम् ।
यद्भवति तद् घुणाक्षरमतोऽस्फुटाभ्यां किमेताभ्याम् ॥ —*BrSpSi* XI 51

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CHAPTER XIII

Brahmagupta and Astronomical Instruments

The Twenty second Chapter of the *Brahmasphuṭasiddhānta* is known as the *Yantrādhyāya* or a chapter on instruments There is a description of seventeen types of time reckoning instruments (*Kāla yantra*)[1]

1 Dhanuryantra—Bow instrument

2 Turyagolaka yantra—Quadrant (one fourth sphere)

3 Cakra yantra=wheel or circle

4 *Yaṣṭi yantra—a pole or staff instrument*

5 Śanku yantra—Gnomon

6 Ghaṭikā yantra—a clock or pot instrument

7 Kapāla yantra—Bowl or potsherd instrument

8 Karttarī yantra—Scissor or knife cutter

9 Pīṭha yantra=Pedastal or seat instrument

10 Salila yantra—Water leveller

11 Brahma or Śāṇa yantra—For describing circles

12 Avalamba Sūtra—Threads with plumbs (Plumb lines)

13 Karṇa or chāyā-karṇa—A [illegible]re of squares for diagonals

14 Chāyā or śaṅku-chāyā—Sundial

1 सप्तदश कालयन्त्राण्यतो धनुस्तुर्यगोलकं चक्रम् ।
यष्टिः शङ्कुर्घटिका कपालकं कर्तरी पीठम् ॥
सलिलं भ्रमोऽवलम्बः कर्णश्छाया दिनार्धमर्कोऽक्षः ।
नतकालज्ञानार्थं तेषां संसाधनान्यथो ॥

—*BrSpSi* XXII 5-6

15. Dinārdha yantra—Midday measure instrument.
16. Arka yantra—Sun-instrument.
17. Aksa or Palānśa yantra—Small degree measure arc instrument.

Salila yantra is used for levelling; since a liquid such as water seeks its own level, it can be utilised to know whether a surface has been levelled or not.[1] Bhrama or *Śāṇa* is used for drawing circles. Avalambaka or plambline is used for adjusting vertical line. *Karṇa* is used in connection with angles and diagonals. From *Salila* (no. 10) to the last (no. 17); these eight are used for adjustments and are basically important.

The *dhanuryantra* is used for *nata* and *unnata* kāla *ghaṭikās*.

On the *paridhi* or the circumference of the *cakra-yantra* are indicated the twelve *rāśis*. ending up to Mīna (XXII. 18). Brahmagupta has described the *yaṣṭi yantra* and shown how it could be used to give time at different parts of the day, and from its shadow *dṛgjyā* and other characteristics can be calculated. This instrument can also be used for ascertaining the solar-lunar differences, and for fixing up the directions. It can be used for determining various heights and altitudes.

The *karttarī yantra* is of the shape of a pair of scissors with two semi-circular blades, fastended to a string at the centre; at the centre is fixed a pin or a pole which casts shadows.

Setting up of the Gnomon

Here it would be interesting to describe the setting of a gnomon, which K.S. Shukla has given in details while commenting on the *Mahābhāskarīya* (IV. 1):

> After having tested the level of the ground by means of water, draw a neat circle with a pair of compasses (*karkaṭa*) (At the centre of that circle, set up a vertical gnomon). The gnomon should be large, cylindrical, massive, and tested for its perpendicularity by means of four threads with plumbs (*avalmbaka*) tied to them.

1. सलिलेन समं साग्रं भ्रमेण वृत्तमवलम्बकेनोर्ध्वम् ।
तिर्यक् कर्णेनान्यैः कथितैश्च नव प्रवक्ष्यामि ॥ —*BrSpSi* XXII. 7.

Bhāskara I in his commentary on the *Āryabhaṭīya* tells us that there was a difference of opinion amongst astronomers in his time regarding the shape and size of gnomon (also called style) Some astronomers prescribed a gnomon with its one-third in the bottom of the shape of a prism on a square base (*caturasra*) one-third in the middle of the shape of a cow's tail (*go pucchākāra*) and one third at the top of the shape of a spear-head (*Śalākāra*) and some others prescribed a square prismoidal (*samacaturasra*) gnomon The followers of Āryabhaṭa I, he informs us prescribed the use of a broad (*pṛthu*) massive (*guru*) and large (*dīrgha* cylindrical gnomon made of excellent timber and free from any hole, a scar or knot on its body In the above stanza Bhāskara I prescribes this last kind of gnomon the other two kinds he proves in the commentary to be defective and so he rejects them

For getting the shadow-end easily and correctly the cylindrical gnomon was surmounted by a fine cylindrical iron or wooden nail fixed vertically at the centre of the upper end The nail was taken to be longer than the radius of the gnomon so that its shadow was always seen on the ground

Certain writers, Bhāskara I tells us in the commentary prescribed a gnomon of half a cubit (=12 *aṅgulas*) in length and having twelve divisions But according to Bhāskara I (although it was the usual custom) there was no such hard and fast rule The gnomon could be of any length and any number of divisions The gnomon should however, be large enough so that the rings of graduation on the gnomon may be clearly seen on the shadow A broad and massive gnomon was preferred because it was unaffected by the wind

Brahmagupta describes gnomon which at the bottom is two *aṅgulas* wide, pointed as a needle 12 *aṅgulas* in length, and full of holes from the basic circular part to the pointed extremity (*BrSpSi* XXII 39)

As regards testing the level of the ground Bhāskara I observes

> When there is no wind place a jar (full) of water upon a tripod on the ground which has been made plane by means of eye or thread and bore a (fine) hole (at the bottom of the jar) so that the water may

> have continuous flow. Where the water falling on the ground spreads in a circle, there the ground is in perfect level, where the water accumulates after departing from the circle of water, it is low, and where the water does not reach, there it is high (Bhāskara's Commentary on the *Āryabhaṭīya*, II 13).

After the ground was levelled, a prominently distinct circle was drawn on the ground as stated in the text (*MBh* III. 1) In the time of Sankaranārāyana (869 A D), there it seems that all lines were drawn on the ground with sandal paste (candanākṣodārdra) The above circle having been thus drawn and coated with sandal paste, another small concentric circle was drawn with the radius of the gnomon The gnomon was then placed vertically with the periphery of its base in coincidence with that circle The gnomon was thus set up exactly in the middle of the bigger circle The verticality of the gnomon was tested by means of four plumb lines hung on the four sides of the gnomon

Gnomon Used for Finding the Directions

The rule in this connection has been described by Brahmagupta in *BrSpSi*. III. 1. The same rule in other words has been described by Bhāskara I in *MBh* III 2 In the *Vāsanā Bhāṣya*, Pṛthūdaka Svāmī describes the details of determining the directions The level of the ground is ascertained by means of water and a gnomon of 12 angulas is set up Find out two points where the shadow of the gnomon enters into and passes out of the circle Bhāskara prescribes drawing out a fish figure with these points The thread line which goes through the mouth and tail of the fish figure indicates the north and south directions with respect to the gnomon Brahmagupta says that if the Sun is on the eastern side then where the shadow point enters circle (in the forenoon) that point would be the west, and the point where it emerges out (in the afternoon) is the east.

As the Sun moves along the ecliptic, its declination changes By the the time the shadow moves between the forenoon and afternoon points as given above the Sun traverses some distance of the ecliptic and, so, theoretically speaking its

declination gets changed It follows. therefore that the East West line in the above determination is not the true position of the actual East West line *Brahmagupta* (628 A D) *was the first Hindu astronomer who prescribed the determination of the East West line with proper allowance for the change in the Sun's declination* (Shukla) The details of the method intended by him have been supplied by his commentator Pṛthūdaka Svāmi (860 A D)

Bhāskara and Brahmagupta both give another method of determining directions (*BrSpSi* III 2 *MBh* III 3) With the three points (at the ends of the three shadows of the gnomon) corresponding to (any three) different times (in the day) draw two fish figures (each with two of the three points) in acco dance with the usual method From the point of intersection of the lines passing through the mouth and tail (of the two fish figures) determine the north and south directions. (*MBh* III 3)

> have continuous flow Where the water falling on the ground spreads in a circle there the ground is in perfect level . where the water accumulates after departing from the circle of water, it is low , and where the water does not reach there it is high (Bhāskara s Commentary on the *Āryabhaṭīya*. II 13)

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Brahmagupta in his rule is more precise

> The point where the lines passing through the two fish figures, which are drawn by means of three shadow ends (of the gnomon) intersect each other is for places in the northern hemisphere the south direction (if the midday shadow falls to the north of the foot of the gnomon) If the midday shadow falls towards the south of the foot of the gnomon it is the north direction (*BrSpSi* III 2)

This rule is obviously based on the assumption that the locus of the end of the shadow of the gnomon is a circle In fact the locus for places whose latitude is less than (90°—the obliquity of the ecliptic) this locus is a hyperbola

Brahmagupta has made numerous uses of gnomon He and Bhaskara for example both give the rules for finding the latitude and colatitude and the z nith distance and altitude of the Sun by finding out the length of the shadow and the length of the gnomon (*BrSpSi* III 10 *MBh* III 5) also rule for the determination of the latitude with the help of the Sun's meridian zenith dis ance and declinat on (*BrSpSi* III 13, *MBh*

III. 17); also rule for finding the Sun's altitude (*BrSpSi.* III. 27; *MBh.* III. 24) (The Sun's altitude for the night has been called by Brahmagupta as *pātāla-śaṅku*, *BrSpSi.* XV. 9).

Golayantra or Armillary Sphere

The first mention of the Golayantra or the armillary sphere is in the *Aryabhaṭīya* (Golapāda. 22)[1] which was a uniformly round circle made of wood or of bamboo and which was of uniform weight or density alround. It was levelled with mercury, oil or water. A śalākā or pin (or rod) was fixed in it in the south-north direction. Its description from the commentary *Bhaṭadīpikā* of Paramādīśvara is given here :

> A sphere of wood, uniformly round on all sides and with uniform density, and also light is made to revolve round an iron axis fixed north-south without friction (oil may be introduced to avoid friction). To the backside of the sphere is fixed a *nalaka* full of water which has the length equal to the circumference of the sphere; and which has a hole at the bottom.
>
> Now a thread, connected to the hook of the wooden ball (on the top side) passing over another small ball (in the same axis of the wooden ball) is attached to the mercury lobe by its other end. The mercury lobe is placed on the level of water and water is allowed to flow through the bottom hole and with water mercury lobe also goes down. The time in which the above hook of ball comes to bottom (180°) is noted. The experiment is repeated with oil. The use of this mechanism is to revolve the ball by water or oil[1]

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1. काष्ठमयं समवृत्तं समन्तात्सम गुरु लघुं गोलम् ।
पारत तैलजलैस्तं भ्रमयेत्स्वधिया च कालसमम् ॥ *Arya. IV. 22*

काष्ठमयं वंशादि काष्ठेन निर्मितं समवृत्तं सर्वतोवृत्तं समन्तात्सम गुरु सर्वावयवेषु समं गुरुत्वं यथा भवति तथा कृतं । लघुमगुरुं एवं भूतं गोलं कृत्वा पारतादिभिः स्वधिया च कालसमं भ्रमयेत् । अयमर्थः । भूमिष्ठ दक्षिणोत्तरस्तम्भयोरुपरि गोलमध्ये दक्षिणोत्तरशलाकाया अग्रे स्थापयेत् । गोलदक्षिणोत्तराच्छिद्रे च तैलेन सिञ्चेत् यथा निस्संगो गोलो भ्रमति । गोलस्याधस्तो गोलपरिधितुल्य दैर्घ्यं छायाच्छिद्रं अम्बुपूर्णं नलकं निदध्यात् ततो गोलस्योपरि कीलकं निधाय तस्मिन् सूत्रस्यैकं अग्रं बद्ध्वा त्रिभुजन्यपट्टलकूटेन प्राङ्मुखं नीत्वा तदग्रं पारतपूर्णं नलकं जलपूर्णे नलके निदध्यात् ततो नलकस्याध

[Cont. on Page 337]

In the Arabic epitome of the Almagest entitled *Tahriru'l mejesti* the armillary sphere *Za ul halk* is thus described

> Two equal circles are placed at right angles, the one representing the ecliptic the other the solstitial colure Two pins pass *throught the poles of the ecliptic* and two other pins are placed on the poles of the equator On the first two pins are suspended a couple of circles moving the one within the other without the first mentioned circles and representing two secondaries of the ecliptic On the two other pins a circle is placed, which encompasses the whole instrument and within which the different circles turn it represents the meridian Within the inner secondary of the ecliptic a circle is fitted to it in the same plane and turning in it This is adapted to measure latitudes To this internal circle two apertures or sights opposite to each other and without its plane are adapted like the sights of an instrument for altitudes The armillary sphere is complete when consisting of these six circles The ecliptic and secondaries are to be graduated as minutely as may be practicable. It is best to place both secondaries as by some directed, within the ecliptic (instead of placing one of them without it) that the complete revolution of the outer secondary may not be obstructed by the pins at the poles of the equator The meridian likewise should be doubled or made to consist of two circles the external one graduated and the internal one moving within it. Thus the pole may be adjusted at its proper elevation above the horizon of any place The instrument so constructed consists of seven circles

> It is remarked that when the circle representing the meridian is placed in the plane of the true meridian so

छिद्रं विहितं कुर्यात् तेन जलं निस्रवति। नलकस्य जलमधो गच्छति। तद्वशाच्च तत्रस्थमन्त्रा पारतपूर्णो गुरुत्वज्जलेन सहाधो गच्छद् गोलं प्रत्यङ्मुखमाकर्षति। एवं त्रिंशद्घटिकाभिरर्धसमितं यथा कालं भवति गोलस्य चार्धं भ्रमति तथा स्वबुद्ध्या जलनिस्रावो योज्य । इति।

अमृतस्रावयोगेन कालभ्रमणसाधनम्।
गणबीजसमाकृष्टं गोलयन्त्रं प्रकल्पयेत्॥ *SaS: XIII, 16-17*

that it cuts the plane of the horizon at right angles and one of the poles of the equator is elevated above the horizon conformably with the latitude of the place then the motions of all the circles round the poles represent the motions of the universe

After rectifying the meridian if it be wished to observe the Sun and Moon together the outer secondary of the ecliptic must be made to intersect the ecliptic at the Sun s place for that time and the solstitial colure must be moved until the place of intersection be opposite to the Sun Both circles are thus adjusted to their true places or if any object but the Sun be observed the colure is turned until the object be seen in its proper place on that secondary referred to the ecliptic the circle representing the ecliptic being at the same time in the plane of the true ecliptic and in its proper situation Afterwards, the inner secondary is turned towards the Moon (or to any star intended to be observed) and the smaller circle within it bearing the two sights is turned until the Moon (or to any star intended to be observed) and the smaller circle within it, bearing the two sights is turned until the Moon be seen in the line of the apertures The intersection of the secondary circle and ecliptic is the place of the Moon in longitude and the arc of the secondary, between the aperture and the ecliptic, is the latitude of the Moon on either side (North or South) (From Colebrooke s *Miscellaneous Essays*)

The same instrument as described by Montucla from the text of Ptolemy (1 3 c 2) consists of six circles first a large circle representing the meridian next four circles united toge ther representing the equator ecliptic and two colures and turning within the first circle on the poles of the equator lastly a circle turning on the poles of the ecliptic, furn ith and nearly touching on its concave side the the ecliptic

The armillary sphere described by the A differs therefore from Ptolemy s in omittin

equinoctial colure, and adding an inner secondary of the ecliptic, which as well as the meridian is doubled

According to Lalande the astrolobe of Ptolemy from which Tycho Brahe derived his equatorial armillary, consisted only of four circles two placed at right angles to represent the ecliptic and solstitial colure a third turning on the poles of the ecliptic and serving to mark longitudes, and a fourth within the other three furnished with sights to observe celestial objects and measure their latitudes and longitudes.

Whether the ancient Greeks had any more complicated instrument formed on similar principles, and applicable to astronomical observations is perhaps uncertain We have no detailed description of the instrument which Archimedes is said to have devised to represent the phenomena and motions of the heavenly bodies nor any sufficient hint of its construction nor does Cicero s account of the sphere exhibited by Posidonius suggest a distinct notion of its structure

Among the Arabs no addition is at present known to have been made to the armillary sphere between the period when the Almagest was translated and the time of Alhazen who wrote a treatise of optics, in which a more complicated instrument than that of Ptolemy is described Alhazen s armillary sphere is stated to have been the prototype of Tycho Brahe s, but neither the original treatise nor the Latin translation of it are procurable and one is therefore unable to ascertain whether the sphere mentioned by the Arabian author resembled that described by Indian astronomers At all events, says Colebrooke he is more modern than the oldest of the Hindu writers.

Here we give the literal translation of the passage on armillary sphere or *Golayantra* occurring in the *Sūrya-Siddhānta*

> Let the astronomer frame the surprising structure of the terrestrial and celestial spheres.
> Having caused a wooden globe to be made (of such size) as he pleases to represent the Earth with a staff for the axis passing through the centre and exceeding the globe at both ends, let him place the supporting hooks, as also the equinoctial circle.
> Three circles must be prepared (divided for signs and degrees) the radius of which must agree with the

respective diurnal circles in proportion to the equinoctial the three circles should be placed for the Ram (*Meṣa*) and following signs respectively, at the proper declination in degrees N or S , the same answer contrariwise for the Crab (*Karkaṭa*) and other signs In like manner three circles are placed in the southern hemisphere for the Balance (*Tulā*) and the rest and contrariwise for Capricorn (*Mṛga*) and remaining signs Circles are similarly placed on both hoops for the asterisms in both hemispheres as also for *Abhijit* and for the *Seven Ṛṣis Agastya Brahmahṛdaya* and other stars

In the middle of all these circles is placed the equinoctial At the intersection of that and supporting hoops the distant from each other half the signs the two equinoxes should be determined and the two solstices, at the degrees of obliquity from the equinoctial and the the places of the Ram (*Meṣa*) and the rest in the order of the signs, should be adjusted by the strings of the curve Another circle thus passing from equinox to equinox is named the ecliptic and by this path the Sun illuminating worlds for ever travels The Moon and other planets are seen deviating from their nodes in the ecliptic to the extent of their respective greatest latitudes (within the zodiac)[1]

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1 भूगोलकस्य रचना कुर्यादाश्चर्यकारिणीम् ।
अभीष्टं पृथिवीगोलं कारयित्वा तु दारवम् ॥
[illegible] तु मध्यगं मेरु [illegible] विनिर्गतम् ।
आधारकक्ष्याद्वितयं कक्ष्यां वैषुवतीं तथा ॥
भगणाङ्गुलैः कार्या दलितास्तिस्र एव ता ।
अहोरात्रार्धकर्णैश्च तत्प्रमाणानुमानत ॥
क्रान्तिविक्षेपभागैश्च दलिता दक्षिणोत्तरा ।
स्वैरन्त्यपक्रमे कार्या मेषादीनामपक्रमात् ॥
कृत्वा मकरादेस्तास्तु कक्ष्याणीनां विपर्ययात् ।
[illegible] नां भगणानां विलोमत ॥
द्युम्नगोलाभिना कुर्यात् कक्ष्याधारद्वयोपरि ।
अन्योन्यभागसंस्थाना [illegible] ॥
[illegible] प्रकल्पयेत् ।
मध्ये वैषुवती कक्ष्या सर्वासामेव संस्थिता ।

[Cont on page 341]

The author of the *Sūrya Siddhānta* then proceeds to notice the relation of the great circles before mentioned to the horizon and observes that whatever place be assumed for the apex of the sphere the middle of the heaven for that place is its horizon. He concludes by showing that the instrument may be made to revolve with regularity by means of a current of water and hints that the appearance of spontaneous motion may be given by a concealed mechanism for which quicksilver is to be employed. There is a hint of secrecy also in one of the lines, and it has therefore been stated that the construction and the mechanism of working should be learnt under the guidance of a teacher.

How to Observe Places of Stars

Details are not available in this connection. The *Sūrya-Siddhānta* only hints that the astronomer should frame a sphere and examine the apparent longitude and latitude (*sphuṭavikṣepa and sphuṭadhruvaka*). The commentators however describe the manner of making the observation. They direct a spherical instrument (*Golayantra*) to be constructed as described above. This instrument is very much similar to the armillary sphere. An additional circle graduated for degrees and minutes is directed to be suspended on the pins of the axis as pivots. It is named as *Vedhavalaya* or intersecting circle and appears to be a circle of declination. After noticing this addition to the instrument the instructions proceed to the rectifying of the *Golayantra* or armillary sphere which is said to be placed so that the axis shall point to the pole and the horizon be true by a water level.

The instrument being thus placed, the observer is instructed to look at the star *Revatī* through a sight fitted to an orifice at the centre of the sphere and having found the star to adjust by it the end of the sign Pisces on the ecliptic. The observer is then to look through the sight, at the *yoga* star of *Aśvinī* or at

some other proposed object and to bring the moveable circle of declination over it The distance in degrees from the intersection of this circle and ecliptic to the end of *Mina* or Pisces is its longitude (*dhruvaka*) in degrees and the number of degrees on the moveable circle of declination from the same intersection to the place of the star is its latitude (*vikṣepa*) North or South

The commentators have rightly remarked that the latitude so found is *sphuṭa* or apparent being the place intercepted between the star and the ecliptic on a circle passing through the poles but the true latitude (*asphuṭa*) is found on a circle hung upon the poles of the celestial sphere as directed in another place (From Colebrooke s Paper on the Indian and Arabian Divisions of the Zodiac. *Miscellaneous Essays* Vol II 324-326)

For the details of the *Golayantra* readers are requested to refer to the description in the *Siddhānta Śiromaṇi* of Bhāskara II

Reference

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